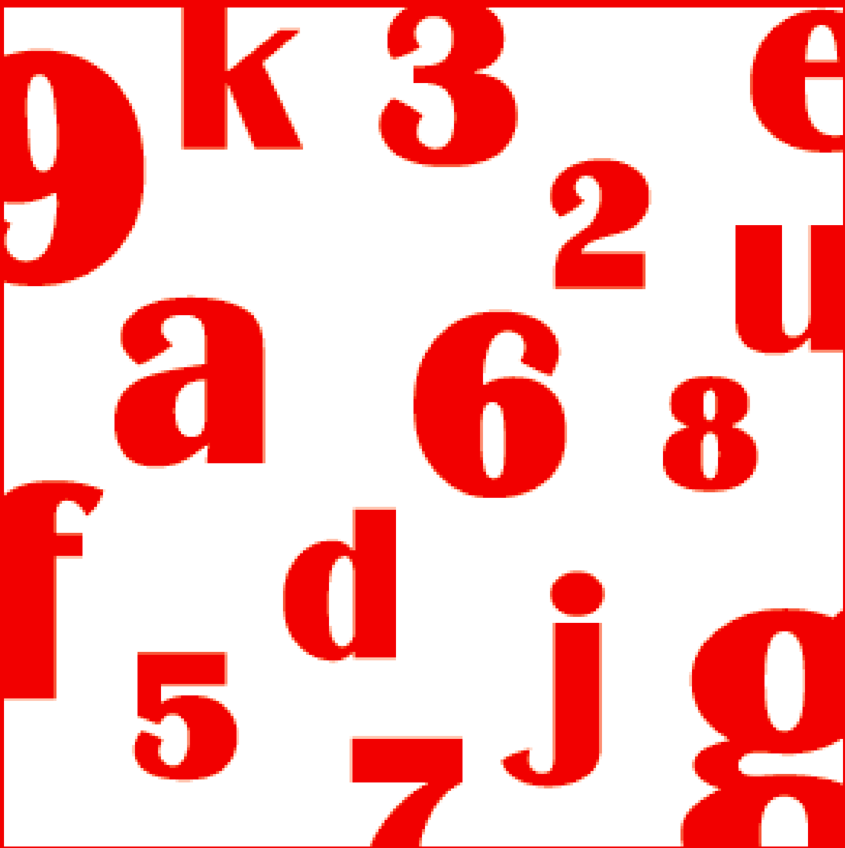
PERMUTATION & COMBINATION
PROBABILITY



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Published in India by

Takshzila
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www.takshzila.com

MRP: Rs. 350

Permutation & Combination

Permutation & Combination – it invariably evokes groans from a lot of students. And ironically the topic is just about counting, counting the number of ways in which certain event can happen. Since the counting can extend to a large number, we would fall short of fingers to count with and hence certain rules are framed. So P & C is essentially about rules to help us count.

Fundamental Rules of Counting

Rule of *AND*

If event A can happen in m different ways and event B can happen in n different ways, event *A and B* can happen in $m \times n$ ways.

If I have 3 different shirts and 2 different trousers, the total number of different ways in which I can wear a shirt and a trouser is 3 × 2 = 6 ways.

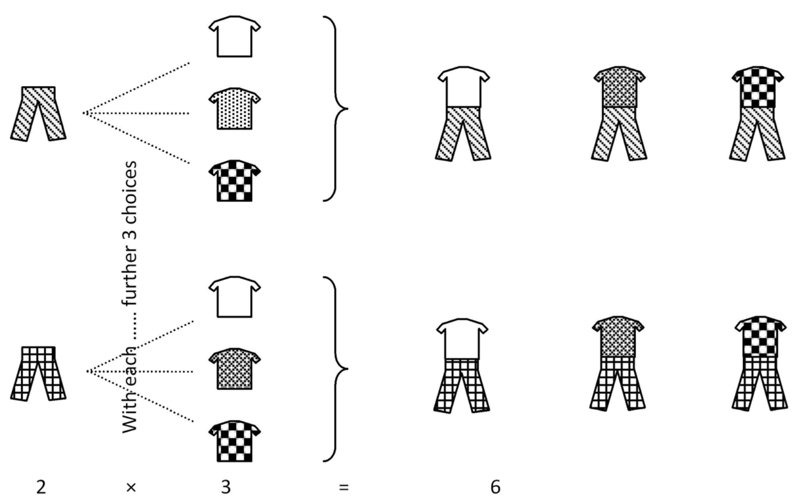
While the above would appear obvious, please read the explanation given in the box and acquaint yourself with the 'tree of possibilities' because this will be very handy in further questions ……

Explanation:

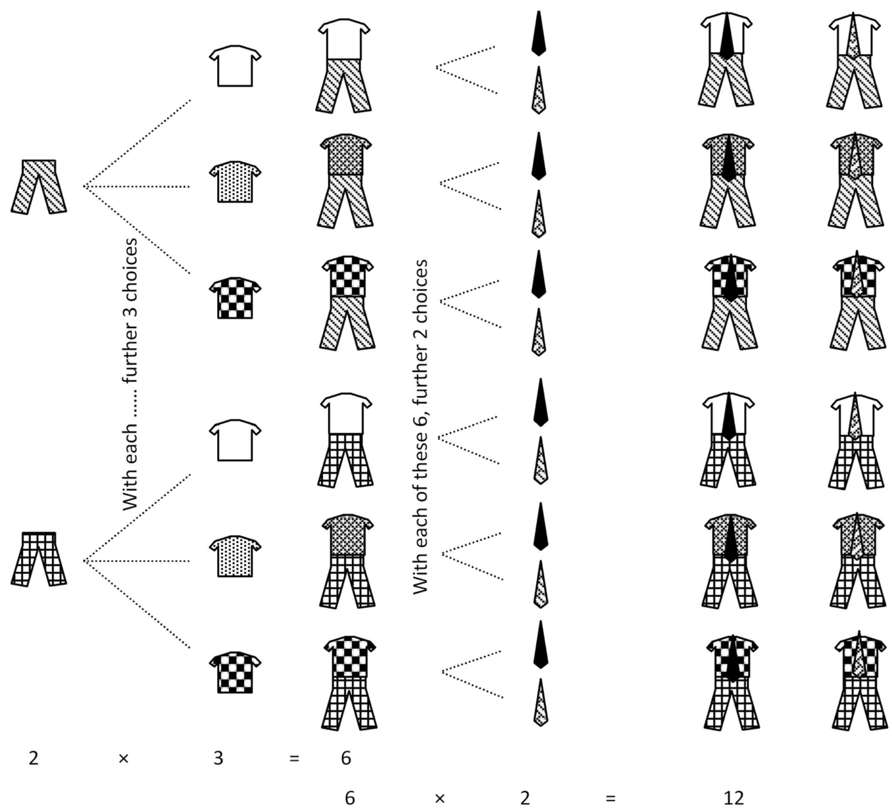
Since we have 2 trousers, we can pick any of the two and thus we have 2 choices of selecting a trouser.

For each of the above choices, *there are further 3 choices* to select a shirt.

Pay attention to the underlined and italicized language, it will be used often. And it should immediately suggest that we would have to multiply 2 and 3 (*for each … further 3 choices*).



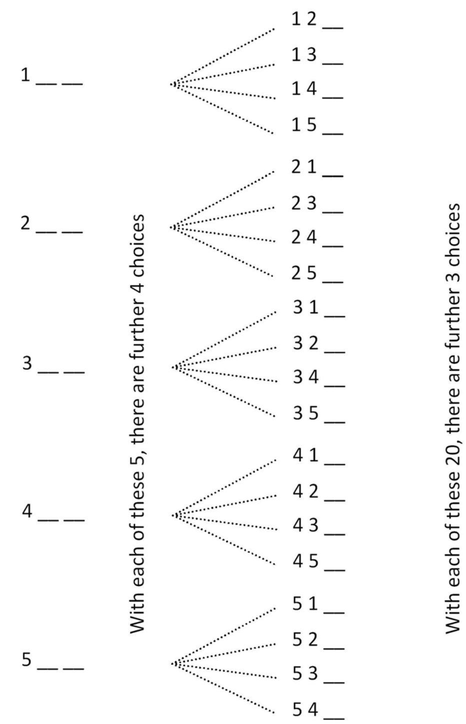
The rule is not limited to just two events. In the above case if I also had 2 different ties and I had to also wear a tie, then just with shirts and trousers, we already had 6 different possibilities. Now *with each of above* 6 possibilities, *there are further* 2 ways to select a tie and thus in all the possibilities would be 6 × 2 = 12.



The above would look obvious, because it is shirts, trousers, ties and we regularly mix and match. What about questions like "How many 3 digit numbers can one form using 1, 2, 3, 4 and 5, each digit being used only once?"

Well this question is EXACTLY same as the above. Just as we have a fixed place for wearing shirts, trousers and ties, in a three digit number also we have three places – units, tens and hundreds. And instead of 2 trousers, 3 shirts and 2 ties, here we have to use the digits 1, 2, 3, 4 and 5 (the only difference being that any of the digits can go in any place unlike the fact that you cannot wear a shirt in place of trouser).

Starting with the hundreds position, it can be filled in 5 ways, any of 1, 2, 3, 4, or 5. In each of these 5 ways, one of the digits out of the 5 given digits has been used and cannot be used again. Thus, f*or each of these above* 5 ways, *there are further* 4 ways in which the ten's digit can be filled – any of the remaining 4 digits could be filled in the ten's position. Thus we get 5 × 4 = 20 different ways in which the hundred's and ten's digit can be filled. See the diagram.



Further in each of these 20 ways, two digits out of the available five digits would have been used and cannot be used again in filling the units place. Thus *for each of the above* 20 ways, *there are further* 3 ways to fill the units place. Thus, the total possible numbers formed will be 20 × 3 = 60.

Using the rule of *AND*: the hundreds position can be filled in 5 ways; having filled the hundreds position, the tens position can be filled in 4 ways; and having done this, the units position can be filled in 3 ways. Since we have to fill the hundreds position *AND* the tens position *AND* the units position, the total number of ways is 5 × 4 × 3 = 20.

Is P&C difficult?

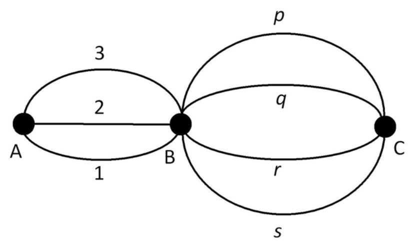
The above example is given here just to explain that P&C is nothing but basic counting rules. Just as we 'combine' different shirts, trousers and ties to dress up differently, similarly we can combine different digits to 'dress' up a number.

If the context is removed, P&C is just rules for counting. If there is a rectangular grid of dots, 10 rows and 8 columns, and we need to find the number of dots in the grid, we don't sit and count each dot, we know that there will be 10 × 8 = 80 dots. This is also a rule of counting.

Similarly, when we have to fill 3 places and we have 5 objects with us, each occupying only one place, the counting rule logic learnt above states this can be done in a total of 5 × 4 × 3 = 60 ways.

The above rule in notation is written as ${}^{5}P_{3}$. If we just learn the formula by rote, P&C can appear tough and opaque. But if we learn the fundamental rules of counting and not worry about the notation, it would be as easy as counting is.

E.g. 1: There are 3 ways to travel from A to B and 4 ways to travel from B to C.



1. In how many different ways can one travel from A to C?

2. In how many different ways can one travel from A to C and come back to A?

3. In how many different ways can one travel from A to C and come back to A travelling on a different road between each of A to B and B to C while going and while coming back?

1. From A to B one can choose a way in 3 different ways and from B to C one can choose a way in 4 different ways. Since one has to choose a way from A to B *AND* B to C, the total number of different ways is 3 × 4 = 12.

Using simple logic, one would be able to identify the 12 ways as:

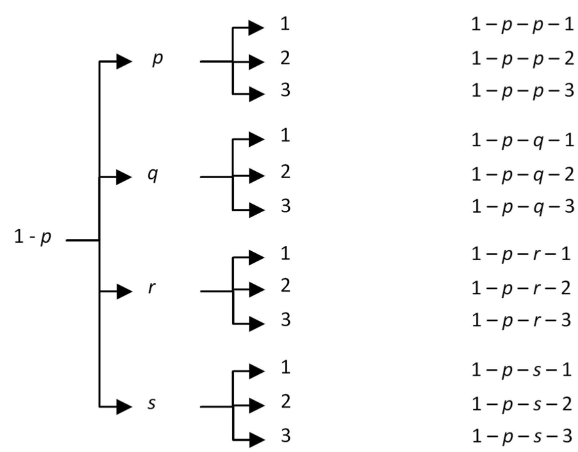
1-*p* 1-*q* 1-*r* 1-*s* 2-*p* 2-*q* 2-*r* 2-*s* 3-*p* 3-*q* 3-*r* 3-*s*

2. A to B can be travelled in 3 ways; B to C in 4 ways; back C to B in 4 ways; and back B to A in 3 ways. Thus total number of ways is 3 × 4 × 4 × 3 = 144.

Logically, f*or each of the above* 12 ways of going from A to C, *there are further* 12 ways of returning from C to A.

Thus the total number of ways of the round trip will be 12 × 12 = 144.

While all 144 cannot be listed here, if we go from A to C by 1-*p*, then while returning we can travel by any of the 12 ways listed above …



Such 12 combinations exists for each of the 12 onwards journey.

3. From A to B we can travel by any of the 3 ways; from B to C by any of the 4 ways; on way back from C to B by any of the 3 ways (the way taken from B to C cannot be taken again); from B to A by any 2 way (the way taken from A to B cannot be taken again).

Thus, total number of ways is 3 × 4 × 3 × 2 = 72 ways.

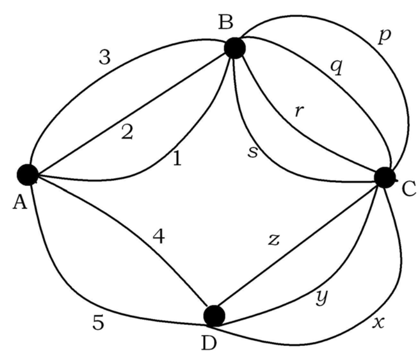
Rule of *OR*

If event *A* can happen in *m* different ways and event *B* can happen in *n* different ways, event *A or B* can happen in $m + n$ ways.

When to use Rule of *OR*

The rule of *OR* is used in cases where there are two (or more) distinct ways in which an event can happen and each of the two (or more) cases need to be handled independently.

E.g. In the example of travelling from A to C, let's say the road network was as shown below:



And we are supposed to find the number of ways in which we can travel from A to C.

If we start with – there are 5 ways to proceed from A (viz. 1, 2, 3, 4, 5), then we would not be able to process further. It would be wrong to say that the total number of ways would be 5 × 4 or 5 × 3 or 5 × 7. After the first leg of journey, which can be done in 5 ways, there are not 4 ways (or 3 ways or 7 ways) for *each of the earlier* 5 ways.

In this case we would need to break the event of travelling from A to C in two distinct cases one via B and one via D. Thus I could travel A-B-C *or* A-D-C.

A-B-C can be travelled in 3 × 4 = 12 ways and A-D-C can be travelled in 2 × 3 = 6 ways.

Since I can travel on only one of these two paths (A-B-C *or* A-D-C), the total number of ways is 12 + 6 = 18. (Remember in this case *for each of the* 12 ways from A-B-C, there ARE NOT *further* 6 ways. Travelling along these two routes is not sequential, one after the other, it is a case of *or*)

E.g. 2: How many natural numbers can be formed using any or all of the digits 1, 2, 3, 4, each digit being used maximum once.

The only difference between this question and the example explained earlier is that in this question the number of digits in the number to be formed is not specified. Thus we can form a single digit number *or* a two-digit number *or* a three-digit number *or* a four digit number. We cannot form a number with digits being more than 4 since each of the given number can be used just once.

Number of single digit numbers that can be formed: 4

Number of two-digit numbers that can be formed: 4 × 3 = 12 (the ten's position can be filled with any of the four digits and *for each of these* 4 ways, the unit's digit can be filled *further in* 3 ways).

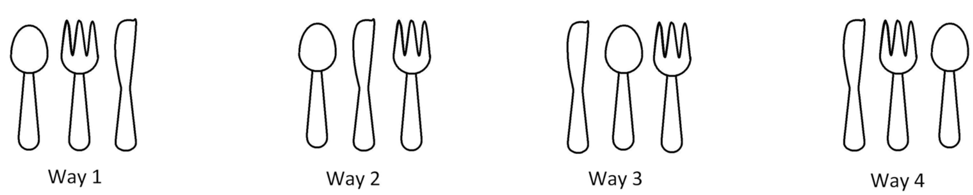
Number of three-digit numbers that can be formed: 4 × 3 × 2 = 24

Number of four-digit numbers that can be formed: 4 × 3 × 2 × 1 = 24

Thus, the total possible natural numbers that can be formed = 4 + 12 + 24 + 24 = 64. (Now we are adding and not multiplying because we can form a single digit *or* two-digit *or* three-digit *or* four-digit)

Arrangements (Permutation)

Arrangements are characterised by 'specific way in which objects are arranged'. The word 'arranged' suggests a specific 'order' among the objects being arranged. E.g. if I have to arrange a fork, a knife and a spoon on the table, each of the following is a different arrangement:



Each of the above depicted way is different is because of 'order' of placing the three objects is different. When we say 'order' we refer to the relative placement of the objects among themselves.

Each of way 1, way 2 and way 3 are different because each has a different object in the middle. Though way 1 and way 4 has the same object in the middle, yet they are different because in way 1, the spoon is to the left of the fork and in way 4 the knife is to the left of the fork.

The above example is just to make you understand what we mean by 'arrangements' or the importance of 'order'. Arranging objects would thus mean placing objects in specific 'position'.

Are these ALL the possible ways of arranging the three objects? It should easily be possible for you to come up with more ways and thus these are not the only way. So what is the total number of possible ways of arranging the three objects among themselves?

We have three positions and three objects. The first position can be filled in three ways with any of the fork, knife or spoon occupying it. Having filled this, *for each of the 3 ways*, the second position can be *further filled in 2 ways*, with either of the two remaining objects. Having filled this, the third position can be filled in only one way with the object remaining. Since we have to place object in first position *and* second position *and* the third position, total number of ways is $3 \times 2 \times 1 = 6$ ways.

Thus, arrangement of objects in positions primarily uses the same funda as Rule of *AND*.

It does not matter what we are arranging

In the example we had to arrange fork, knife and spoon. The same funda is used whether we had to arrange digits of a number, alphabets or people.

Forming a three-digit number using the digits 1, 2, 3 is same as arranging the objects 1, 2 and 3 in three specific positions viz. hundreds, tens and units.

Arranging the alphabets of CAT in all possible ways is also the same as arranging three objects C, A and T in three positions.

Seating five people in a row for a photograph is also the same as arranging 5 objects, the five different people, in a row having 5 positions.

Thus, for Maths, it makes no difference if we are arranging knife, fork, spoon, etc or digits or alphabets or people. The funda remains the same whatever the context is. It's just that each of the context gives rise to different conditional arrangements. E.g. while arranging numbers we could have a condition that number has to be even or number has to be divisible by 3. While arranging people such conditions do not have any meaning. In the context of arranging people we can have conditions like two people want to sit together or they do not want to sit together.

In this case we had to arrange three objects in three places. What if the number of objects was more than the number of positions? We have already solved this situation when we were forming a three-digit number using the digits 1, 2, 3, 4 or 5, each digit used once.

It's obvious that not all the objects can be arranged. The first position can be filled in 5 ways, *for each of these 5 ways*, the second position *can be further filled in 4 ways*. *For each of the earlier ways* of filling the hundreds and tens position, the units position *can be further filled in 3 ways*. Thus, total number of ways of arranging is $5 \times 4 \times 3 = 60$ ways. Thus the reasoning remains the same.

Defining ^nP_r

As seen earlier,

the number of ways of arranging 3 objects out of 3 objects is 3 × 2 × 1

the number of ways of arranging 3 objects out of 5 objects is 5 × 4 × 3

By the same logic,

the number of ways of arranging 3 objects out of 11 objects would be 11 × 10 × 9

the number of ways of arranging 6 objects out of 9 objects would be 9 × 8 × 7 × 6 × 5 × 4

To generalise the above, the number of ways of arranging *r* objects in a row out of *n* distinct objects would be $\underbrace{n\times(n-1)\times(n-2)\times\ldots\ldots}_{\text{total of } r \text{ factors}}$. This expression is denoted as ^nP_r.

To calculate the value of ^nP_r, one would necessarily do $\underbrace{n\times(n-1)\times(n-2)\times\ldots\ldots}_{\text{total of } r \text{ factors}}$. But then this is not the ideal way to write a formula (because we need to write the text 'total of *r* factors'). To come up with a proper formula for ^nP_r look at the following working. It takes help that the product $n \times (n-1) \times (n-2) \times \ldots\ldots$ resembles $n!$, though the product is truncated, in a very creative way ……

$$^{11}P_3 = 11\times10\times9 = \frac{11\times10\times9\times8\times7\times\ldots\ldots\times1}{8\times7\times\ldots\ldots\times1} = \frac{11!}{8!} = \frac{11!}{(11-3)!}$$

$$^{9}P_6 = 9\times8\times7\times6\times5\times4 = \frac{9\times8\times7\times6\times5\times4\times3\times2\times1}{3\times2\times1} = \frac{9!}{3!} = \frac{9!}{(9-6)!}$$

Thus, $^nP_r = \dfrac{n!}{(n-r)!}$

A few permutations that should be known directly without using the formula are …

$^nP_1 = n$ (Since we have to just choose 1 object, it could be any of the given *n* objects and can be done in *n* ways)

$^nP_n = n!$ (The first position can be filled in *n* ways; second in $(n-1)$ ways; third in $(n-2)$ ways and the n^{th} position in just 1 way, the last object left. Thus the number of ways is $n \times (n-1) \times (n-2) \times \ldots\ldots \times 1 = n!$

To sum up, the number of ways of arranging *r* objects in a row out of *n* distinct objects is denoted by ^nP_r and is given by the formula $\dfrac{n!}{(n-r)!}$ and its value is calculated as $\underbrace{n\times(n-1)\times(n-2)\times\ldots\ldots}_{\text{total of } r \text{ factors}}$.

Use fundamental rule of counting rather than ^nP_r

Please don't use ^nP_r indiscrimately. A few conditions to use it are:

- can be used only for arrangements in a row or where every position is distinct.
- can be used only when all the *n* objects are distinct
- can be used only in situations where an object can come in only one position i.e. 'repetitions are not allowed'

Further since it is just a notation to write the expression in short and the funda behind the formula is the basic Rule of AND, we strongly suggest that you use the basic Rule of AND to work through the problems.

It would appear surprising that almost all arrangement problems can be solved using the basic rules of counting and one need not get bogged down by ^nP_r.

Arrangement of digits

Forming numbers, as seen above, is same as 'arranging digits' and the funda to be used in the basic Rule of *AND*. However while forming numbers, there can be many conditions like forming even numbers, forming numbers greater than a given number, forming numbers that are divisible by a given number etc. All the following examples show how to handle situations with such conditions……

E.g. 3: Using digits 1, 2, 3 and 4, how many 3 digit numbers can be formed if repetition of digits is not allowed?

While this question may appear very easy since we have already discussed this, do pay attention to the explanation in the box since it makes a few observations which will be useful to us later …

The number of ways in which the 1st place, 2nd place, 3rd place can be filled is 4, 3, and 2 respectively. Thus the total possible numbers that can be formed is $4 \times 3 \times 2 = 24$ different numbers can be formed. This is same as $^4P_3$, in notation.

Two important observations

1. When there are no conditions, it does not matter which position we start filling with.

Look at the following two different ways – in one we start filling the hundreds position and in other we start with tens position ……

(For each of these 12 ways, there are further 2 choices)

1 _ _	1 2 _	1 2 3	1 2 4
	1 3 _	1 3 2	1 3 4
	1 4 _	1 4 2	1 4 3
2 _ _	2 1 _	2 1 3	2 1 4
	2 3 _	2 3 1	2 3 4
	2 4 _	2 4 1	2 4 3
3 _ _	3 1 _	3 1 2	3 1 4
	3 2 _	3 2 1	3 2 4
	3 4 _	3 4 1	3 4 2
4 _ _	4 1 _	4 1 2	4 1 3
	4 2 _	4 2 1	4 2 3
	4 3 _	4 3 1	4 3 2

Filling hundreds place first and then tens place

(For each of these 12 ways, there are further 2 choices)

_ 1 _	_ 1 2	3 1 2	4 1 2
	_ 1 3	2 1 3	4 1 3
	_ 1 4	2 1 4	3 1 4
_ 2 _	_ 2 1	3 2 1	4 2 1
	_ 2 3	1 2 3	4 2 3
	_ 2 4	1 2 4	3 2 4
_ 3 _	_ 3 1	2 3 1	4 3 1
	_ 3 2	1 3 2	4 3 2
	_ 3 4	1 3 4	2 3 4
_ 4 _	_ 4 1	2 4 1	3 4 1
	_ 4 2	1 4 2	3 4 2
	_ 4 3	1 4 3	2 4 3

Filling tens place first and then units place

In both the ways, we end up with the same 24 numbers formed.

2. Another important observation is that each of the digit appears in each of the position, equal number of times. E.g. the digit 4 comes in the hundreds position 6 times. And similarly each of the digits 1, 2 and 3 also come exactly 6 times in the hundreds position. Not just this, each of the digit appears exactly 6 times in each position – hundreds, tens, units.

Both the above observations are true only if there are no conditions attached. See example 5 or 6 for a comparison and significance of the above observations.

E.g. 4: Using digits 1, 2, 3 and 4, how many 3 digit numbers can be formed if repetition of digits is allowed?

Since a digit can be re-used, each of the 3 positions can be filled in 4 ways and thus the total possible numbers that can be formed is $4 \times 4 \times 4 = 64$.

In this example we could not use ${}^{n}P_{r}$ because repetitions are allowed.

E.g. 5: Using digits 1, 2, 3, 4, 5, 6 and 7, how many 5 digit <u>EVEN</u> numbers can be formed, without repeating any digit?

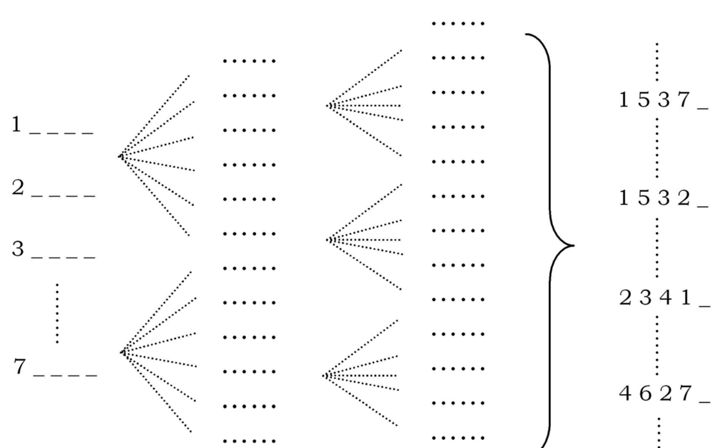
In this question there is a condition on the digits that can occupy the units position. In questions like these, when there is a condition on a certain position, we would NECESSARILY have to start by filling this position first i.e. satisfying the condition.

The units place could be filled in with only 2, 4 or 6 i.e. in 3 ways. Having filled the units place, four more places have to be filled and we have 6 digits remaining. This can be done in a total of $6 \times 5 \times 4 \times 3 = 360$ ways (in notation this will be ${}^{6}P_{4}$).

For each of the 3 ways of filling the units place, *there are further 360 ways* to fill the remaining 4 places. Thus the total number of 5 digit even numbers that can be formed is $3 \times 360 = 1080$.

Why do we have to begin with satisfying the condition first?

Let's consider that we start filling positions from left to right i.e. from the leading position onwards. The leading four places can then be filled in $7 \times 6 \times 5 \times 4 = 840$ ways. These are all the possible ways and in these we have considered every possibility of a particular digit appearing in the leading four places. See the diagram for a few of these ways ……



All possible 840 ways of filling the leading 4 digits

Out of all the possible ways four particular possibilities are written in the right most column.

With the first of these, 1537_, the unit digit can be filled in 3 ways (2 or 4 or 6) and thus 3 more numbers can be formed with these being the leading 4 digits.

With the second of these, 1532_, since 2 is already used up, the unit digit can be filled in only 2 ways (4 or 6).

With the third of these, 2341_, since 2 and 4 are used up, the only way the units digit can be filled is one way viz. with a 6.

And with the last particular number highlighted, we cannot form any even number because each of 2, 4, 6 are used up and we cannot re-use them.

Thus, each of these 840 ways of filling the leading four digits do not result in 'equal' number of further ways in which the unit digit can be used. Thus we cannot consider all the 840 ways as one pattern and proceed forward using the Rule of *AND*. Many cases would have to be considered and this would increase our work manifold.

This is the reason why we start by satisfying the condition first.

Even if we fill any one other position and then proceed to satisfy the given condition, it will lead to problems. E.g. If leading digit is 2, i.e. the number formed is of the type 2 _ _ _ _, the number of ways the units digit can be filled is only two ways (either a 4 or a 6), whereas with the leading digit being 1 i.e. in numbers of the form 1 _ _ _ _, the number of ways the units digit can be filled will be three ways (2, or 4 or 6).

The only way to proceed is to satisfy the given condition first. The above problem does not occur if digits can be repeated. In this case, we can start filling any position first. The answer to this question if repetition of digits is allowed will be 7 × 7 × 7 × 7 × 3 = 7203.

E.g. 6: Using digits 0, 1, 2, 3, 4 and 5, how many four digit numbers can be formed using each digit only once?

When 0 (zero) is being used

Whenever 0 can be used and one has to form, say a four-digit number, please be careful that 0 cannot be placed in the leading position else it will not be a four-digit number.

Thus, while it appears that there is no condition given in the question, there is an implicit condition that 0 cannot appear in the leading position. And as learnt above, since there is a condition, we will have to start by satisfying this condition first. If we fill in other positions (apart from leading position in this case) then at end we would not have a fixed number of ways in which the leading position can be filled for all the earlier cases.

So that 0 does not appear in the leading position, we have to start by filling the thousands place. It can be filled only in 5 ways using any one of 1, 2, 3, 4 or 5. After filling this place, we are left with 5 digits (including zero) and three places to fill and this can be done in 5 × 4 × 3 ways (0 can appear in any of the positions). Thus total number of required numbers that can be formed is 5 × 5 × 4 × 3 = 300.

E.g. 7: Using digits 2, 3, 4, 5, 6 how many four digit numbers can be formed such that they are divisible by 3? Repetition of a digit is not allowed.

The divisibility rule of 3 says that the sum of digits should be divisible by 3. Also we have to use any four out of the 5 given digits since it is a four digit number. So first let's check a number formed by which 4 digits is going to be divisible by 3.

If we use 2, 3, 4 and 5 (i.e. not use 6), the sum of digits will be 14 and thus the number will not be divisible by 3.

If we use 2, 3, 4 and 6 (i.e. not use 5), the sum of digits will be 15 and thus the number will be divisible by 3.

If we use 2, 3, 5 and 6 (i.e. not use 4), the sum of digits will be 16 and thus the number will not be divisible by 3.

If we use 2, 4, 5 and 6 (i.e. not use 3), the sum of digits will be 17 and thus the number will not be divisible by 3.

If we use 3, 4, 5 and 6 (i.e. not use 2), the sum of digits will be 18 and thus the number will be divisible by 3.

Thus only two sets of 4 digits can be used such that the number is divisible by 3 viz. (2, 3, 4 and 6) or (3, 4, 5 and 6)

With *each* of these sets, we can form $^4P_4 = 4!$ i.e. $4 \times 3 \times 2 \times 1 = 24$ numbers and thus a total of 48 numbers divisible by 3 can be formed.

E.g. 8: Using digits 1, 2, 3, 4, 5, 6 and 7 (without repetition), find the number of distinct 4 digit numbers that can be formed for each of the following conditions:

1. greater than 3000

2. even

3. even number greater than 3000.

The first two cases should be very easy as they are exactly similar to earlier examples…

1. The thousand's digit can be filled in 5 ways (any one of 3, 4, 5, 6 or 7). Now one is left with 6 digits and has to arrange 3 of them. This can be done in $^6P_3 = 6 \times 5 \times 4$ ways. Thus a total of $5 \times (6 \times 5 \times 4) = 600$ such numbers can be formed.

2. The unit's digit can be filled in three ways (2, 4 or 6). Now we are left with 6 digits and have to arrange 3 of them. This can be done in $^6P_3 = 6 \times 5 \times 4$ ways. Thus a total of $(6 \times 5 \times 4) \times 3 = 360$ such numbers can be formed.

3. This case is a difficult one and unlike all previous examples. In this case there are conditions on both ends: the thousand's digit should be 3 or greater AND the unit digit should be one of 2, 4 or 6.

$$\underbrace{\_\_\_}_{3,\,4,\,5,\,6,\,7} \quad \_\_\_ \quad \_\_\_ \quad \underbrace{\_\_\_}_{2,\,4,\,6}$$

So the thousands digit can be filled in 5 ways, units place can be filled in 3 ways. Having done this we are left with 5 digits and two places to be filled with. So is the answer $5 \times (5 \times 4) \times 3 = 300$?

This answer is wrong. When there are two conditions, there is an interaction between the two conditions, satisfying one condition affects the possibilities for satisfying the other condition.

If two conditions, which to satisfy first?

If we start with units digit, the units digit can be filled in three ways.

Having filled the units position, in how many ways can we fill the thousands position?

5 ways? Not necessarily. We might have used the digit 4 or 6 in filling the unit's place and so the digit may not be available for us to be filled in the thousand's place. And then the thousands position can be filled in only 4 ways.

Then, the answer should be 4 ways, right? Again not really. It is quite possible that the unit digit was filled with a 2 and thus we have all of 3, 4, 5, 6 or 7 i.e. 5 of them to fill the thousand's place.

Thus, after filling the unit's place, the answer to "in how many ways can the thousand's place be filled" would *depend* on how the unit place is filled. Whenever such a dependency comes in, its best to make use of an *or* condition...

And the same problem will be encountered if we start from the thousands digit ......

The thousands digit can be filled in 5 ways. Having done this ......

if the thousand place was filled with a 3 or 5 or 7, then there would be 3 ways to fill the unit's digit.

if the thousand place was filled with a 4 or 6, then there would be only 2 ways to fill the unit's digit.

Thus, it does not matter which end you start with. The only way to tackle this is using an *OR* ......

If we start with the units digit, we need to break the three possibilities, 2, 4 and 6, into two groups such that the number of possibilities for the thousands digit is unique for each group.

The unit digit could be 2 OR (4, 6)...

$$\_\_\_\_ \quad \_\_\_\_ \quad \_\_\_\_ \quad \underbrace{\_\_\_\_}_{2} \quad OR \quad \_\_\_\_ \quad \_\_\_\_ \quad \_\_\_\_ \quad \underbrace{\_\_\_\_}_{4,\,6}$$

Considering the case that the units digit is 2:

The units digit can be filled in only 1 way, with a 2. Having filled this, all of 3, 4, 5, 6 and 7 are available for the thousands place and thus it can be filled in 5 ways. Next we are left with 5 digits and two of which have to be arranged i.e. can be done in 5 × 4 ways. Thus total possible numbers with units digit being 2 will be 5 × (5 × 4) × 1 = 100

Considering the case that the units digit is 4 or 6:

The units digit can be filled in 2 ways. Having filled this, for each of these case, the digits available for the thousands place are 3, 5, 7 and (4 or 6, whichever is not used) i.e. there are further 4 possibilities. Next we are left with 5 digits and two of which have to be arranged i.e. can be done in 5 × 4 ways. Thus total possible numbers with unit's digit being 4 or 6 will be 4 × (5 × 4) × 2 = 160

Thus total number of even numbers greater than 3000 that can be formed will be 100 + 160 = 260.

If we started with the thousands digit we would have to consider the following two cases:

Case 1: When thousands digit is 3, 5 or 7. In this case there would be 3 possibilities for the units digit.

Case 2: When thousands digit is 4 or 6. In this case there would be 2 possibilities for the units digit.

The answer obtained would be the same.

What if repetition of digits was allowed in this question?

If repetition of digits is allowed, it becomes very very easy, because then satisfying one condition does not affect the possibilities for the other condition. In this question, the thousands digit can be filled in 5 ways, the units place in 3 ways (irrespective of what is filled in thousands place) and the other two places, each can be filled in 7 ways. Thus the answer would have been 5 × 7 × 7 × 3 = 735

Exercise

1. Using all digits from 0 to 9, how many 5 digit numbers can be formed? Repetition is allowed.

   (1) 10^5 (2) $9 \times 8 \times 7 \times 6 \times 5$ (3) 90,000 (4) 9^5

2. How many 6 digit numbers can be formed using 1, 2, 3, 4, 5 and 6 such that all odd positions (from left end) are odd digits and all even positions are even digits? Repetition of digits is not allowed.

   (1) 72 (2) 36 (3) 720 (4) 360

3. Using digits 0, 1, 2, 3 how many natural numbers can be formed such that no digit is repeated in the number?

   (1) 18 (2) 24 (3) 36 (4) 48

4. How many 6 digit numbers can be formed using the digits 1, 2, 3, 4 and 5 (repetitions allowed) such that the number reads the same from left to right or from right to left (e.g. 134431)?

   (1) 6 (2) 12 (3) 24 (4) 36

5. How many 4 digit numbers can be formed using the digits 1, 2, 3, 4 such that atleast one digit is repeated?

   (1) 256 (2) 24 (3) 232 (4) 64

6. Using digits 0, 1, 2, 3 how many four digit numbers divisible by 4 can be formed? Repetition of digits is allowed.

   (1) 36 (2) 48 (3) 60 (4) 72

7. Using 0, 1, 2, 3 and 4 (without repetition), how many four digit even numbers can be formed?

   (1) 72 (2) 60 (3) 56 (4) 48

8. Using 1, 2, 3, 4 and 5, without repetition, how many four digit numbers can be formed such that the digits of the number, from left to right, are in ascending order?

   (1) 5 (2) 120 (3) 96 (4) 72

9. How many four digit numbers can be formed that have the digit 5 used exactly once in them?

   (1) 3600 (2) 3700 (3) 2916 (4) 3033

10. What is the sum of all the four digit numbers that are formed using each of the digits 1, 2, 3 and 4 exactly once?

    (1) 266,640 (2) 711,040 (3) 66,660 (4) 66,666

Arranging people in a row

As stated already arranging people in a row is also very much like forming numbers because for Maths it hardly makes any difference if you are placing digits in places or persons in positions. The only difference being that instead of conditions like number being even or divisible by 3, we could have different conditions like two or more people wanting to be together or not wanting to be together or being in alternate position.

E.g. 9: In how many ways can 6 people be arranged in a row when two of them are adamant about standing at either extreme position?

As learnt earlier, we should satisfy the given conditions first. The two people, say *A* and *B*, can be placed at the extreme ends in two ways viz. *A* at the left end and *B* at right or *A* at the right end and *B* at left (the right end is different from the left end). Now we are left with 4 people and 4 places, which can be arranged in $4 \times 3 \times 2 \times 1 = 24$ ways. Thus the total number of ways of arranging is $2 \times 24 = 48$ ways.

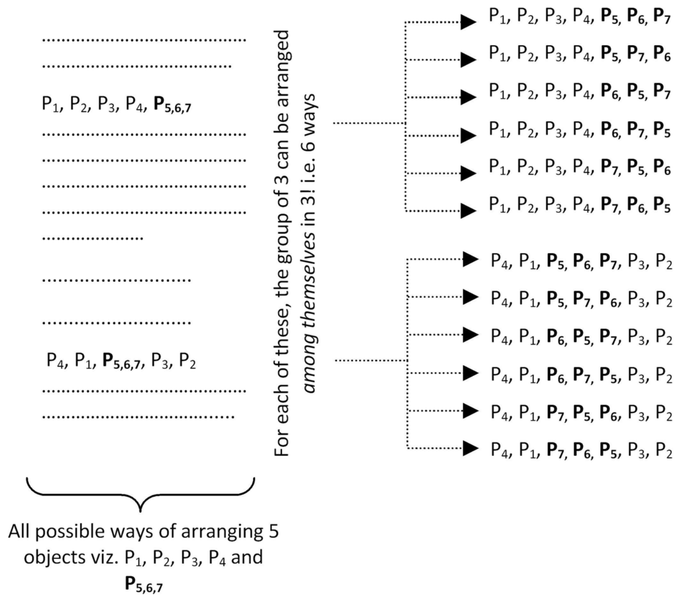
E.g. 10: In how many ways can 6 people be arranged in a row when two of them are adamant about standing at either extreme position and another two persons are adamant about standing in the middle?

The two persons who want to be at the extremes can be placed in 2 ways. Similarly the persons who want to be in the middle (i.e. the 3^{rd} and 4^{th} place) can be placed in 2 ways. We are left with 2 people and 2 places which can be filled in 2 ways. Thus in all, the total number of ways of arranging with the given condition is $2 \times 2 \times 2 = 8$ ways.

Condition for persons to be together:

Let us understand this with an example: There are 7 people who have to be arranged in a row. If three of them want to be together, in how many ways can the arrangement be done?

Since the three persons cannot be separated, consider them as a one (imagine tying all three with a rope and consider this group as one person when placing them in the row). Now we have to arrange 4 + 1 i.e. 5 persons in a row which can be done in ${}^5P_5 = 5! = 120$ ways. But for *each of these 120 ways*, the three persons which are considered as one can be *further arranged among themselves*, giving rise to different arrangements. The three persons can arrange themselves in $3! = 6$ ways. Thus for *each of the 120* arrangements, there are *further 6 different arrangements* and the total number of arrangement will be $120 \times 6 = 720$. If it is not understood why we are multiplying see the figure given below (where the three persons insisting on being together are P_5, P_6 and P_7).



Approach for "have to be together"

The way to proceed is to consider all those who have to be together as one object and go ahead with the overall arrangement (arranging this one object and the others who are not particular about their position). Now for each of these arrangements, further arrange those who insist on being together *among themselves*.

E.g. 11: There are 4 boys and 4 girls to be arranged in a row. If all the 4 girls have to be together, in how many ways can the arrangement be done?

Considering all the four girls as one unit, we have to arrange 5 units in 5 places and this can be done in 5! ways. Now one of the units is of 4 girls, who can be arranged amongst themselves in 4! ways. Thus the total number of ways of arranging is $5! \times 4!$.

E.g. 12: In a group of 10 people, 4 speak on French, 3 speak only Spanish and the rest speak both French and Spanish. In how many ways can the 10 people be arranged in a row such that all those who speak only French are together and so are all those who speak only Spanish.

Considering the 4 who speak only French as 1 unit and the 3 who speak only Spanish also as 1 unit, we have to arrange 1 + 1 + 3 (rest) = 5 objects in a row. This can be done in 5! ways.

Now for each of the above 5! arrangement, the four French speaking persons can be arranged among themselves in 4! ways. Thus, the total number of ways in which the 10 people can be arranged considering the 3 who speak only Spanish as 1 unit is $5! \times 4!$.

Again for each of above 5! × 4! arrangements, the three Spanish speaking persons can be arranged among themselves in 3! ways.

Thus the answer would be 5! × 4! × 3!.

Conditions for persons not being together

Again let's understand this case with an example: There are 6 boys and 4 girls to be arranged in a row. If no two girls should stand together, in how many ways can the arrangement be done?

No two girls are together v/s all girls are not together

A beginner's error is to solve the above question in the following manner:

Finding the number of ways in which all the four girls are together ......

Considering the 4 girls as one unit, the 6 boys and this one unit can be arranged in 7! ways. For each of these ways, the unit of 4 girls can be arranged among themselves in 4! Ways. Thus total number of arrangements where all 4 girls are together is 7! × 4!

Next the total number of arrangements of the 6 boys and 4 girls with no condition is 10!.

Since we need 'no two girls together', the answer would be subtracting all the possible ways where girls are together from the total possible ways i.e. 10! – 7! × 4!.

This is incorrect.

We have subtracted all cases where 'all four girls are together'. Thus the answer found is for the number of ways where 'all four girls are not together'. This is different from 'no two girls are together'.

After the subtraction we will still be left with cases where two girls may be together or three girls may be together. E.g. the arrangement B_1 B_2 G_1 G_2 B_3 B_4 G_3 B_5 G_4 B_6 would still be counted in the above answer, whereas it should not be counted in the correct answer.

NOTE: The above approach of subtracting the cases when they are together can be used ONLY when the number of objects that do not have to be together is exactly 2. In that case if we subtract all the cases where the two of them are together from the total possible ways of arranging, we would get all the cases where the two of them are not together.

In questions where no two units should be together, we should first arrange the *other* units and then place the units that should not be together, one each in the spaces between the other units arranged.

Since no two girls should be together, arrange the *others*, 6 boys in this case, first. The six boys can be arranged in 6! ways. Now there are seven places created between the boys and at either end, as shown below for two random cases of arranging the boys:

__ B_1 __ B_2 __ B_3 __ B_4 __ B_5 __ B_6 __

__ B_1 __ B_5 __ B_6 __ B_3 __ B_2 __ B_4 __

For <u>each</u> of these 6! ways ......

...... if not more than one girl is placed in each of the 7 spaces, it will ensure that no two girls will be together. But we have 7 spaces and only 5 girls. It just means that two of the spaces will be vacant. An empty space would mean the boys will be adjacent, which is acceptable because nothing is mentioned about boys being or not being together.

Thus, 5 girls are to be placed and there are 7 available spaces. This can be done in $^7P_5 = 7 \times 6 \times 5 \times 4$ ways. Alternately using rule of *AND*, the first girl can be placed in 7 ways, second girl can be placed in 6 ways, third girl can be placed in 5 ways and fourth girl in four ways.

Thus, the total number of ways of arranging is $6! \times (7 \times 6 \times 5 \times 4)$.

But then there aren't 13 chairs?

Please do not worry about the vacant spaces. Do not think that we have 7 + 6 = 13 chairs and we have to arrange 6 boys and 4 girls in them. This will change the question. There are no chairs.

The vacant spaces are just a creation of ours. Think of the arrangement as a dynamic arrangement, when a girl is placed between two boys, just think the two boys move away a bit. If there is no girl between two boys, the two boys are together (there is no space between them). The arrangement with our created vacant spaces can easily be thought as a continuous arrangement as shown below:

$G_1\ B_1\ \_\ B_2\ \_\ B_3\ \_\ B_4\ G_2\ B_5\ G_3\ B_6\ G_4$ is same as $G_1\ B_1\ B_2\ B_3\ B_4\ G_2\ B_5\ G_3\ B_6\ G_4$

The arrangements are different because of "relative position among themselves" and not about who sits in which chair.

E.g 13: There are 4 boys and 4 girls to be arranged in a row. If no two girls should stand together, in how many ways can the arrangement be done?

Approach for "no two should be together"

Ignoring all the objects of which no two should be together, arrange the rest of the objects first.

Now place the objects of which no two should be together in the gaps between any two objects in the earlier arrangements of the rest of the objects.

The four boys can first be arranged in 4! ways. Now there will be 5 places for the 4 girls and they could be arranged in $5 \times 4 \times 3 \times 2$ ways. Thus the total number of arrangements possible is $4! \times 5 \times 4 \times 3 \times 2$.

E.g. 14: In a group of 10 people, 4 speak on French, 3 speak only Spanish and the rest speak both French and Spanish. In how many ways can the 10 people be arranged in a row such that all those who speak only French are together and so are all those who speak only Spanish. Further all people should be able to converse with both their neighbours.

Considering all the French speaking people as 1 unit, denoted by F, and all the Spanish speaking people as 1 unit, denoted by S, we have to arrange F, S, X_1, X_2, X_3 (where X refers to those who can speak both languages)

The condition that each person should be able to converse with both the neighbours implies that *F* and *S* should not be together. Thus, the others i.e. X_1, X_2 and X_3, have to be arranged first and can be done in 3! ways. Now there are 4 spaces created as follows: __ *X* __ *X* __ *X* __. The *F* and the *S* can be arranged in any two of these 4 spaces in $^4P_2$ i.e. 4 × 3 ways.

For each of the 3! × (4 × 3) ways of arranging *F*, *S*, X_1, X_2, X_3 such that F and S are not together ......

......since *F* is a group of 4 people they can be arranged among themselves in 4! ways and

......since *S* is a group of 3 people they can be arranged among themselves in 3! ways.

Thus the answer would be 3! × (4 × 3) × 4! × 3!

Conditions for persons to be alternate

E.g. 15: There are 4 boys and 4 girls to be arranged in a row. If girls and boys have to be alternate, in how many ways can the arrangement be done?

> **Common confusion**
>
> One of the common confusion among students is: how is this case different from e.g. 13 above where no two girls were together?
>
> In the case where no two girls are together, nothing is mentioned about the boys and the boys could be together e.g. the arrangement $G_1\ B_1\ G_2\ B_2\ B_3\ G_3\ B_4\ G_4$ is acceptable because no two girls are together. But this arrangement will not be acceptable if boys and girls have to be alternate.
>
> Boys and girls being alternate automatically means no two girls are together. But the reverse is not true i.e. no two girls are together does not mean boys and girls *have* to be alternate. They may be and it's also possible they may not be.
>
> Thus, girls and boys being alternate is a subset of no two girls being together.

When girls and boys have to be alternate, it would just be either G B G B G B G B *or* B G B G B G B G. In *each* of these ways, there are 4 places for the boys and 4 places for the girls and thus they can be arranged in 4! × 4! in each of these. Thus, the total number of arrangements possible is 2 × 4! × 4!.

> **Number of boys and girls matter**
>
> If there are 5 girls and 4 boys, the only way they can be alternate is G B G B G B G B G. There is no other way.
>
> Also in this case the two cases, "girls and boys being alternate" and "no two girls being together", which are liable to be confused as explained earlier, are exactly the same. (However "no two boys are together" is distinct from the above case, it will include many more arrangements in addition to the above arrangement)
>
> If there are 6 girls and 4 boys, it is just not possible to arrange them such that girls and boys are alternate. Nor is it possible to arrange them such that no two girls are together. (It is possible to arrange them such that no two boys are together)

Exercise

11. For a group photograph, 4 girls and 5 boys have to be arranged in two rows, with all the girls sitting in the first row and all the boys standing in the second row. In how many ways can they be arranged?

    (1) 9!   (2) $2 \times 4! \times 5!$ (3) $4! \times 5!$ (4) $4 \times 4! \times 5!$

12. In how many ways 8 friends be arranged in a row if three of the friends do not want to sit at the extremes?

    (1) 14,400   (2) $8! - 6!$ (3) $8! - 2 \times 6!$ (4) 7,200

13. 4 men and 4 women have to be seated in a row such that all the men are together and all the women are also together. In how many ways can they be seated?

    (1) 576   (2) 24   (3) 48   (4) 1152

14. 4 managers, 2 vice-presidents and 1 president have to be seated in a row for a meeting such that the two vice-presidents sit on either side of the president. In how many ways can they be seated?

    (1) 120   (2) 240   (3) 360   (4) 48

15. In how many ways can 8 cars be parked in 8 parking slots in a row such that there are exactly 4 cars between two specific cars?

    (1) $6 \times 6!$ (2) $3 \times 6!$ (3) $4 \times 6!$ (4) $8 \times 6!$

16. In how many ways can the letters of the word INVESTOR be arranged such that no two vowels are together?

    (1) $8! - 5! \times 3!$ (2) 7200   (3) $8! - 6! \times 3!$ (4) 14400

17. There are 8 rowers on a rowing boat. They have to be seated such that 4 sit on the right hand side and 4 on the left hand side of the boat. If 3 of the rowers insist on sitting on the right side and another 2 of them insist on not sitting on the same side, in how many ways can the rowers be seated?

    (1) 720   (2) 480   (3) 1152   (4) 576

18. In a group of 9 people there are 4 Gujrathis and 5 Marwadis. Of the Gujrathis, 2 are vegetarian and rest are non-vegetarians. Of the Marwadis, 2 are vegetarian and rest are non-vegetarians. They have to be seated along a row. No two Marwadis want to sit together. Further all vegetarians insist on sitting together. In how many ways can the entire group be seated?

    (1) $4! \times 5! \times 4!$ (2) $6! \times 4! - 2 \times 4! \times 5!$ (3) $6 \times 4! \times 5!$ (4) $6 \times 2! \times 2! \times 2! \times 3!$

Circular Arrangement

Circular arrangement (or an arrangement that has characteristics of both linear and circular arrangement) is distinct from linear arrangement. And the method used to tackle them is also different.

Arrangements are relative positioning "*among themselves*"

Consider the following two linear arrangement of A, B, C and D:

A B C D and D C B A

Are the two different or same? In all the above examples we have treated them as different and they are different.

The fact that each of the objects have the same neighbor (or that the ends are occupied by A and D), do not make them the same.

While the neighbours are same, their 'order' has changed. In first arrangement B has A to *its* right and in second arrangement B has A to *its* left. (Consider they are people who are facing out of the page and right and left are not from observer's perspective but from their perspective, throughout this text)

Similarly in first arrangement, A is to the left end of the row whereas in the second A is to the right end of the row. In other words in first arrangement, there is no one to A's right but in second arrangement, A has C to *its* right.

This makes the arrangements different.

Another example: Say there are four vacant chairs and you have to select any one of them to sit in.

— — — —

Is sitting at either of the extremes the same?

If one just considers that both of them are extremes and hence they are same, one is ignoring the fact that one extreme is at right end of the row and one extreme is at left end of the row. If I sit in the first position, there would be no one to my right, but if I sit at the fourth position, there would be someone to my right. Thus, they both are different. Similarly the two chairs in the middle are not the same; one has one chair to its right whereas the other has two chairs to its right.

Thus, in a linear arrangement all the positions are distinct.

Consider arranging A, B, C and D in a row. One of the ways is A B C D.

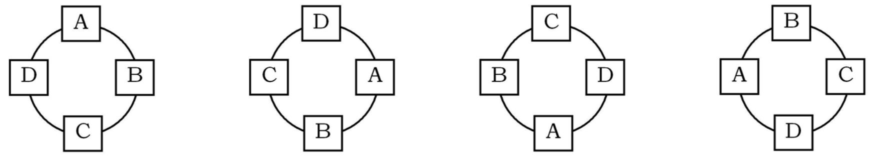
Now if each one gets up and moves one place to the right, we would get the following four arrangements from this arrangement itself

A B C D   D A B C   C D A B   B C D A

If they move again, we would arrive back at the original arrangement viz. A B C D.

Obviously all the four arrangements are different arrangements because different people occupy the extremes and there are numerous other differences related to right and left of each individual.

Consider the same event happening (each one moving one place to the right) in case of a circular arrangement ......



Each of these 4 ways, in a circular arrangement is same. In every aspect *among themselves*.

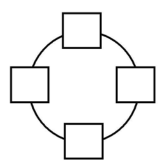
Not only does each individual have the same neighbour, but also the order of neighbours is the same. In all the four arrangements, B has A to *its* right and C to *its* left. (Consider them to be looking inside the circle).

The fact that the north position is taken by different individuals in the four possibilities is resorting to an "external reference" – north. In arrangements, we have to consider differences "among themselves".

Thus, a circular arrangement is definitely distinct from a linear arrangement.
Let's take an example to understand the approach to solve a circular arrangement question.

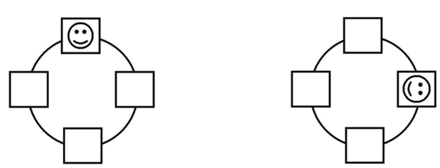
In how many ways can four people be seated around a circular table with four equi-spaced chairs?

Consider a table with four chairs as follows:



Imagine you are the first person to be given the privilege of choosing any position. In how many ways can you select the position you want to sit in? Most often the answer is 4. Think again.

If you are still stuck with 4 as your answer, try to find difference between the following two "apparently different" ways



Without resorting to an external reference, both these ways are the same. Imagine you are the lone occupant on a circular table of 4 and there is no external reference. If you move from one chair to the other, for you the table will still appear exactly the same.

In fact whichever of the 4 chairs, the first occupant sits, all of them is the same. Thus the first occupant can choose his position in only 1 way.

With the first person choosing his seat in 1 way, the second person now has three different ways, could sit opposite to the first person or to his left or to his right. Thus, the second person can choose his position in 3 ways; the third person can choose his chair in 2 ways; and the last person in 1 way. Thus all four of them can sit in 1 × 3 × 2 × 1 = 3! ways.

Different approaches and suggested one

The above could also be understood as follows:

Four people can be arranged in a row in 4! ways. The following picture shows you how this can be used to find the answer if the arrangement had been circular. In the left hand column the 4! ways are arranged such that all arrangements with each one moving one place to his right are written clubbed together ......

A B C D
D A B C
C D A B
B C D A
} Just 1 circular arrangement

A B D C
C A B D
D C A B
B D C A
} Just 1 circular arrangement, different from earlier one (neighbours have changed)

....
....
....
....
....
....

D A C B
B D A C
C B D A
} Just 1 circular arrangement, different from earlier ones (neighbours have changed)

All possible 4! ways

Every carefully chosen 4 linear arrangements will result into just one circular arrangement. Thus if there are 4! linear arrangements, the number of circular arrangements would be $\frac{4!}{4} = 3!$ circular arrangements.

Having explained this approach, we strongly suggest that you approach questions on circular arrangement using the earlier approach of 'just one way for the first person' and then finding the number of ways for each of the successive person. This is necessary because the above approach will become more cumbersome with arrangements having both, linear and circular characteristics.

Consider 8 people to be seated across a circular table. For the 1[st] person though there are 8 chairs, the relative position of each chair is the same and hence he can choose his position in only 1 way. The moment he sits on any of the chairs, there is an internal reference (among themselves) created for the other people. The second person can choose his position in 7 ways, third can choose in 6 ways and so on. Thus, 8 people can be seated across a circular table in $1 \times 7 \times 6 \times 5 \times \ldots \times 1 = 7!$ ways.

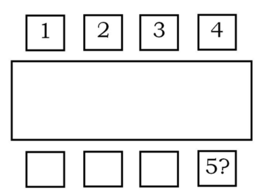
In general, n people can be seated across a circular table in $(n-1)!$ ways.

E.g. 16: 4 boys and 4 girls have to be seated around a circular table such that no two girls are adjacent to each other. In how many ways can they be seated?

Since it is a circular table and no two girls are adjacent to each other, they have to be on alternate chairs and the remaining set of alternate chairs will be occupied by the boys. Now the four boys can be seated in four alternate chairs in a circular arrangement in 3! ways. With the boys seated, there are 4 chairs for the girls and there is already a reference created because of the boys sitting. Thus the 4 girls can be seated in 4 chairs in 4! ways (and not 3!). Thus the total number of ways of seating is $3! \times 4!$.

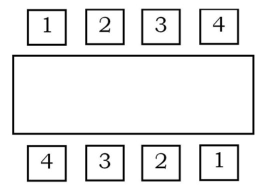
E.g. 17: 8 people have to be seated on a rectangular table with 4 each on the longer sides. In how many ways can they be seated?

In this case, is there just 1 way for the first person to select his position? It should be obvious that each of the positions marked 1, 2, 3 and 4 are distinct ways of sitting, since they are in a row.



Is the position marked 5 yet a new position? For the first person i.e. considering the entire table unoccupied, this position is exactly same as position # 1, in all aspects. Not only are both the positions at ends, they are also at the right end of their respective rows. Each of them have the same number of positions to their left, facing them, etc. Imagine you sitting on a vacant table at either of the position. The table would appear exactly same from each of these positions.

Similar reasoning would help you identify that there are essentially 4 different positions for the first person. In the following figure, every similarly numbered chair is exactly similar in placement and cannot be considered as two choices but as one choice only for the first person …



Thus the first person has only 4 choices to sit and after he has taken any of the chairs, now each chair is distinct because he creates a reference point for other. Once the first person sits in any position, the two similarly marked chairs become distinct, one is on the same side as the side chosen by the first person and the other is on the opposite side. Thus, all the 7 chairs now are distinct and the remaining 7 people can sit in 7! ways.

Thus, the total number of ways all 8 can sit is $4 \times 7!$.

Exercise

19. In how many ways can 10 people be seated across a circular table if

i. There are 11 identical chairs placed equally apart around the table

(1) 12! (2) 11! (3) 10! (4) 9!

ii. If there are 11 distinctly coloured chairs placed equally apart around the table.

(1) 12! (2) 11! (3) 10! (4) 9!

iii. If there are 10 identically coloured chair and 1 chair is distinctly coloured.

(1) 12! (2) 11! (3) 10! (4) 9!

20. 4 men and 4 women have to be seated in a circle such that all the men are together and all the women are also together. In how many ways can they be seated?

(1) 576 (2) 24 (3) 48 (4) 1152

21. 4 managers, 2 vice-presidents and 1 president have to be seated in a circle for a meeting such that the two vice-presidents sit on either side of the president. In how many ways can they be seated?

(1) 120 (2) 240 (3) 360 (4) 48

22. In how many ways can 6 Indians and 8 Americans sit across a circular table with 14 equi-spaced chairs such that no two Indians are sitting next to each other?

(1) $7! \times 5!$ (2) $8! \times 6!$ (3) $7! \times 7!$ (4) $7! \times 6!$

23. In how many ways can 6 couples be seated around a circular table such that each couple is sitting together?

(1) 11! (2) $6! \times 2^6$ (3) $5! \times 2^5$ (4) $5! \times 2^6$

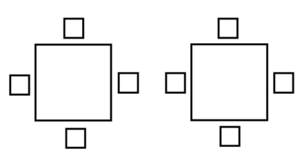
24. In how many ways can 16 people be seated across a square table with 4 persons sitting on each side of the square?

(1) 16! (2) $8 \times 15!$ (3) $4 \times 15!$ (4) 15!

25. In how many ways can 10 people be seated across a rectangular table with 4 on each of the longer sides and one on each of the shorter side of the table?

(1) $5 \times 9!$ (2) $4 \times 9!$ (3) 10! (4) 10!/4

26. In how many ways can eight people be seated across two identical square tables, with one on each side, as shown in the following picture



(1) 8! (2) $2 \times 7!$ (3) $4 \times 7!$ (4) 7!

Arranging Objects when few are identical

Arranging alphabets of a word

In how many ways can the letters of the word NATURE be arranged among themselves?

As commented earlier, maths makes no distinction in arranging digits, people or alphabets. The above question is: in how many ways can 6 distinct objects be arranged among themselves? And the answer as known earlier is 6!. In the same context we could also have any of the conditions learnt earlier like some alphabets being together or not being together, etc.

E.g. 18: In how many ways can the letters of the word "NATURE" be arranged such that all the vowels are together?

There are three vowels viz. A, U and E and three consonants viz. N, T and R. Since the vowels have to be together, considering them as one unit, we have to arrange 4 objects and this can be done in 4! ways. The three vowels can be arranged amongst themselves in 3! ways. And thus the total number of ways of arranging the letter with the vowels being together is 4! × 3!

However the similarity ends the moment the word has the same alphabet repeated. Let's understand the difference and the approach to solve such cases using an example:

In how many ways can the letters of the word REVENUE be arranged among themselves?

This case is different because one letter, E, appears multiple times in the word and hence we have to arrange seven letters of which 3 are identical. Thus $^7P_7$ will be wrong because that is only for arranging 7 distinct objects.

Considering that all the 7 alphabets are distinct, say we are arranging R E_1 V E_2 N U E_3, we can arrange them in $^7P_7$ i.e. 7! ways. In these 7! ways, we have also *arranged* the three E's considering that they are different.

Each arrangement that differs ONLY in the *arrangement* of E's (relative positioning of all the rest of the letters is the same) is counted as distinct arrangements but is the same. E.g. V N E_1 R E_2 E_3 U and V N E_2 R E_1 E_3 U are counted as two different arrangements but essentially are the same arrangement viz. V N E R E E U.

With the same relative positioning of rest of the letters i.e. with V N __ R __ __ U, and arranging E_1, E_2 and E_3, in the blanks, how many arrangements are possible? This is same as arranging 3 distinct objects in 3 places and can be done in 3! i.e. 6 ways. Thus, all these arrangements are the same but have been counted as 6 different ways while getting the answer as 7!.

Thus, if there are *n* distinct ways of arranging the letters of the word REVENUE, each of these *n* distinct way is counted 6 times and the answer arrived at is 7!. Thus, $n \times 3! = 7!$ i.e. $n = \frac{7!}{3!}$. See figure for a more clearer understanding...

....
....
....
V N E_1 R E_2 E_3 U
V N E_1 R E_3 E_2 U
V N E_2 R E_1 E_3 U
V N E_2 R E_3 E_1 U
V N E_3 R E_1 E_2 U
V N E_3 R E_2 E_1 U
....
....
....
U E_1 R E_2 N E_3 V
U E_1 R E_3 N E_2 V
U E_2 R E_1 N E_3 V
U E_2 R E_3 N E_1 V
U E_3 R E_1 N E_2 V
U E_3 R E_2 N E_1 V
....
....
....

Considering all letters distinct, total number of ways = 7!

Correcting the fact that E_1, E_2 and E_3 are not different but the same. Each of these could be arranged among themselves in 3! ways, thus each distinct arrangement is counted 3! times

....
....
V N E R E E U
....
....
....
U E R E N E V
....
....
....

Considering the E's as identical, $= \frac{7!}{3!}$

The same approach would have to be used to arrange the letters of the word ANJANA. The difference here is that there are 2 letters viz. A and N, which appear multiple times in the word.
To start with consider all the 6 letters are distinct letters (assuming the three A's as A_1, A_2 and A_3 and the two N's as N_1 and N_2. Now follow the diagram ......

....
....
....
N_1 A_1 N_2 J A_2 A_3
N_1 A_1 N_2 J A_3 A_2
N_1 A_2 N_2 J A_1 A_3
N_1 A_2 N_2 J A_3 A_1
N_1 A_3 N_2 J A_1 A_2
N_1 A_3 N_2 J A_2 A_1

N_2 A_1 N_1 J A_2 A_3
N_2 A_1 N_1 J A_3 A_2
N_2 A_2 N_1 J A_1 A_3
N_2 A_2 N_1 J A_3 A_1
N_2 A_3 N_1 J A_1 A_2
N_2 A_3 N_1 J A_2 A_1

....
....
....

Considering all letters distinct, total number of ways = 6!

Correcting the fact that A_1, A_2 and A_3 are not different but the same. Each of these could be arranged among themselves in 3! ways, thus each distinct arrangement is counted 3! times

....
....
N_1 A N_2 J A A

N_2 A N_1 J A A
....
....
....

Considering the A's as identical but the N's distinct, total number of ways $= \frac{6!}{3!}$

Correcting the fact that N_1 and N_2 are not different but the same. Each of these could be arranged among themselves in 2! ways, thus each distinct arrangement is counted 2! times.

....
....
N A N J A A
....
....

Considering the A's as identical and also the N's, total number of ways $= \frac{6!}{3! \times 2!}$

While solving the above questions, one just needs to think that one is arranging 6 letters of which 3 are alike and another 2 are also alike and hence the answer is $\frac{6!}{3! \times 2!}$.

> Arranging objects when few of them are alike
>
> The total number of ways of arranging *n* objects of which *m* are identical is $\frac{n!}{m!}$
>
> The logic can extended as follows: If there *n* objects of which *p* objects are of one type, *q* objects of another type, *r* objects of yet another type and all others are distinct, the total number of ways of arranging all the objects is $\frac{n!}{p! \times q! \times r!}$

E.g. 19: In how many ways can the letters of the word "MISSISSIPPI" be arranged?

In the word MISSISSIPPI, there are 11 letters of which the alike ones are four S, four I, and two P. Thus they could be arranged in $\frac{11!}{4! \times 4! \times 2!}$.

E.g. 20: In how many ways can the letters of the word "OPPOSITION" be arranged so that the three O's and the S are together?

Since the three O and S has to be together, let's consider them as a group. Thus we have to arrange (OOOS), P, P, I, T, I, N. These are 7 objects of which two are P and two are I. They can be arranged in $\frac{7!}{2! \times 2!}$ ways.

Next, in the group of 3 O's and S, they can be arranged amongst themselves in $\frac{4!}{3!}$ ways.

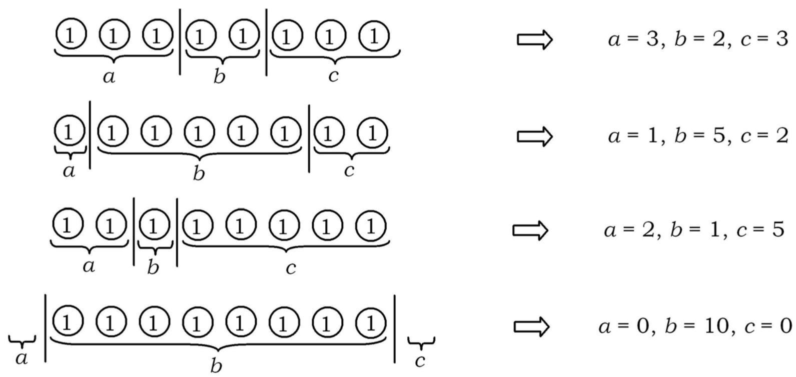
Thus the total number of ways of arranging with the given condition is $\frac{7!}{2! \times 2!} \times \frac{4!}{3!}$

Whole number solution to $a + b + c + \ldots = n$

A unique application of the arranging objects, of which few are identical, is in solving questions like ...

Find the number of whole number solutions to the equation $a + b + c = 8$.

Consider the right hand side, 8, as eight ones or for simplicity as eight 1 Re coins. And then these eight coins have to be distributed among a, b and c. The following figure shows some way of distributing these coins and the equivalent solution of a, b and c.



The above are just a few of the possible solutions. To find all the possible number of solutions, we would need to consider all possible ways of arranging eight 1 Re coins and the two 'bars' that separate the group of eight coins into three groups. Denoting the eight coins as 1 1 1 1 1 1 1 1 and the two bars (partitions) as p, p, we need to arrange 11111111pp in all possible ways. Considering 11111111pp also as a word (though meaningless) similar to the MISSISSIPPI case, the number of ways of all possible arrangements can be found as $\frac{10!}{8! \times 2!}$. (There are a total of 10 objects, 8 are identical to each other and other two are also identical to each other)

In any arrangement of 11111111pp, the number of 1's before the first p is given to a, the number of 1's between the two p's is given to b and the number of 1's after the second p is given to c. Thus, the arrangement 11p11111p1 would result in a solution $a = 2$, $b = 5$ and $c = 1$, whereas the arrangement 111pp11111 will be $a = 3$, $b = 0$ and $c = 5$.

To conclude that the above is fool-proof, we just need to confirm the one-to-one correlation between the number of whole number solution to $a + b + c = 8$ and the number of ways of arranging 11111111pp ......

Every unique way of arranging 11111111pp results in a unique solution and

Each and every solution to $a + b + c = 8$ results into a unique arrangement of 11111111pp

Thus there is no double counting nor is there any under-counting.

Thus, the approach to solve problems similar questions will be as follows:

To find the number of whole number solution to $a + b + c + d + e = 25$, we would need to divide twenty-five 1's into five groups and thus we would need four partitions i.e. we would need to arrange twenty-five 1's and four p's and this can be done in $\frac{29!}{25! \times 4!}$.

Please go through the following solved examples for slight variations in such questions.

E.g. 21: Find the number of natural number solutions to $a + b + c + d = 20$

The difference in this question is that we need natural number solutions and not whole number solutions i.e. none of a, b, c and d can be equal to 0, they have to be atleast 1.

Considering this question as distributing twenty 1 Re coins among a, b, c and d such that each of them get atleast one coin, the best possible way is to give each of a, b, c and d one coin at the start itself. Now we will be left with 16 coins, which have to be distributed among a, b, c and d such that they can even get 0 coins out of these 16 i.e. whole number solution to $a' + b' + c' + d' = 16$.

The number of whole number solution to this equation would be same as the number of ways in which sixteen 1's and three p's have to be arranged i.e. $\frac{19!}{16! \times 3!}$

Consider the following few whole number solutions to $a' + b' + c' + d' = 16$ and their equivalent natural number solution to $a + b + c + d = 20$ ……

$a' = 1$, $b' = 3$, $c' = 7$, $d' = 5$ would mean $a = 2$, $b = 4$, $c = 8$ and $d = 6$

$a' = 0$, $b' = 0$, $c' = 10$, $d' = 6$ would mean $a = 1$, $b = 1$, $c = 11$ and $d = 7$

Thus, each and every whole number solution to solution $a' + b' + c' + d' = 16$ result in a unique natural number solution to $a + b + c + d = 20$.

Thus the number of natural number solution to $a + b + c + d = 20$ would also be equal to $\frac{19!}{16! \times 3!}$.

The same approach can be used in any situation wherever there is a lower limit on the value taken by any of the variables.

E.g. 22: In how many different ways can I purchase a total of 10 fruits if the fruit-seller has mangoes, apples and oranges. Assume the fruit-seller has more than 10 fruits of each variety.

If, m, a and r refer to the number of mangoes, apples and oranges purchased, then it must necessarily be $m + a + r = 10$. Every whole number solution to this equation would result in a different way of purchasing the 10 fruits.

The number of whole number solution to this equation would be same as the number of ways in which ten 1's and two p's have to be arranged i.e.

$$\frac{12!}{10! \times 2!}$$

Exercise

27. In how many ways can the letters of the word “ABRACADABRA” be arranged?

(1) 11! (2) $\frac{11!}{5!}$ (3) $\frac{11!}{5!\times 2!}$ (4) $\frac{11!}{5!\times 2!\times 2!}$

28. In how many ways can the letters of the word “WEDNESDAY” be arranged such that the three vowels are at the two extremes and the middle of the arrangment?

(1) $\frac{3!}{2!}\times\frac{6!}{2!}$ (2) $\frac{9!}{3!\times 6!}$ (3) $\frac{9!}{2!\times 2!}$ (4) $\frac{3!\times 6!}{4!}$

29. In how many ways can the letters of the word MATHEMATICS be arranged such that the two Ms are together, the two As are together and the two Ts are together?

(1) 8! (2) $\frac{11!}{2!\times 2!\times 2!}$ (3) 11! (4) $8!\times 2!\times 2!\times 2!$

30. In how many ways can the letters of the word “ENTERTAINER” be arranged such that no two consonants are together?

(1) $2 \times 5! \times 6!$ (2) $2\times\frac{5!}{3!}\times\frac{6!}{(2!)^3}$ (3) $\frac{5!}{3!}\times\frac{6!}{(2!)^3}$ (4) $\frac{11!}{3!\times(2!)^2}-\frac{6!}{(2!)^3}$

31. How many numbers can be formed using each of the digits 1, 1, 2, 2, 2, 3 and 4?

(1) 4! (2) $\frac{7!}{2!\times 3!}$ (3) $7! - 2! \times 3!$ (4) $7! - 2! \times 3! \times 2!$

32. In how many ways can the letters of the word “ENGINEERING” be arranged such that the relative positions of vowels and consonants do not change?

(1) $\frac{11!}{(2!\times 3!)^2}$ (2) $\frac{5!+6!}{(2!\times 3!)^2}$ (3) $\frac{5!\times 6!}{(2!\times 3!)^2}$ (4) $\frac{5!\times 6!}{2!\times 3!}$

33. Find the number of natural number solutions to $a + b + c + d = 20$ such that ...

i. ... each of a, b, c and d are even. ii. ... each of a, b, c and d are odd.

(1) $^{13}C_3$, $^{11}C_3$ (2) $^{9}C_3$, $^{7}C_3$ (3) $^{9}C_3$, $^{11}C_3$ (4) $^{13}C_3$, $^{7}C_3$

34. How many four-digit numbers are such that the sum of the digits is 10?

(1) $^{13}C_3$ (2) $^{12}C_3$ (3) $^{12}C_3 - 4$ (4) $^{12}C_3 - 1$

35. There are 15 intermediate stations on a railway line from one terminus to another. In how many ways can 4 of these stations be chosen as halts for the train such that between any two of *these* 4 halts there are atleast 2 stations where the train does not halt?

(1) $^{11}C_4$ (2) $^{10}C_4$ (3) $^{9}C_4$ (4) $^{8}C_4$

36. Find the number of natural number solutions to the equation $a+b+c\le 12$.

(1) $^{11}C_2$ (2) $^{12}C_3$ (3) $^{13}C_3$ (4) $^{13}C_2$

Selections (Combination)

So far our task was always to “arrange” objects i.e. to place them in a specific order among themselves.

Sometimes we would be interested in only “selecting” few objects out of the given objects. In this case we just need to “select” and we do not need to “arrange” them in an order. E.g. we need to select 4 students out of 15 students who will represent the college at a quiz or we need to form an academic committee of 3 professors from 10 professors. In this case, who is selected “first”, who is selected “second” and so on does not matter. The words “first” and “second” implicitly implies an “ordering”. What matters in the case of selection is only the composition of the final “group”.

Comparison of Arrangement and Selection

Say we have to arrange 3 people out of A, B, C, D and E in a row.

We have seen that the arrangement A B C and A C B are distinct arrangements because of the relative positioning among themselves.

However if we had to select three people out of the 5 people, both of these “arrangements” would be the SAME “selection” viz. the group {A, B, C}.

In fact all the “arrangements” that are possible with A, B and C, in the case of “selection” would count as only 1 selection. Since A, B, C can be arranged among themselves in 3! i.e. 6 ways, all these 6 ways would be counted as only 1 if it were a case of selection and not arrangement.

A selection different from {A, B, C} would be only when the group composition is different, say {A, B, D} or {A, C, E} or {C, D, E}, etc. Also each of these groups would be counted as only 1 selection, irrespective of the ordering among the group members.

The above distinction should immediately make it clear that the number of selections will be lesser than the number of arrangements because quite a few of the arrangements are counted as just 1 selection. Conversely, with just 1 selection, we can order the group members among themselves and result in many more arrangements from the same selection.

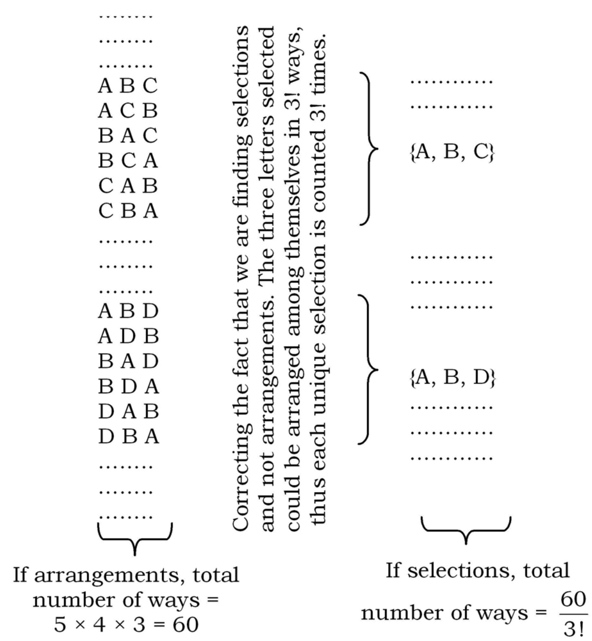
See if you identify the difference between selection (combination) and arrangement (permutation) in similar looking contexts ....

Selection (Combination)	Arrangement (Permutation)
Selecting a team of 11 from 16 probable’s	Drawing a batting line-up of 11 from 16 probable’s
Selecting a committee of 3 from 10 members	Selecting a committee of a president, a vice-president and a treasurer from 10 members
Selecting 3 students out of 10 students who will receive scholarships of same value	Selecting 3 students out of 10 students who will receive three scholarships – one of Rs. 10,000, one of Rs. 5000 and one of Rs. 2000

Number of ways of selecting

Consider the same example as discussed in the above box: selecting 3 people out of A, B, C, D and E. Rather than starting fresh, all over again, let us use the knowledge of arrangements that we have already learnt to find the number of ways in which we can select 3 objects out of 5 distinct objects.

The approach used to find the number of selections is similar to the approach used earlier when we had purposefully arranged identical alphabets and then later corrected by dividing. See the following figure ......



Another way to look at the above figure is to divide the task of arranging 3 people from 5 people into two sequential tasks ......

i. Select 3 people out of the 5 people

ii. Now arrange the 3 selected people among themselves.

Since we know that the number of ways in which 3 people selected can be arranged among themselves is 3!, hence,

(Number of selections) × 3! = (Number of arrangements)

$$\Rightarrow \text{Number of selection} = \frac{\text{Number of arrangements}}{3!}$$

Thus, to find the number of selections, we first find the number of arrangements and then use the fact that certain number of arrangements result in the same selection. We then arrive at the correct answer by dividing the 'number of arrangements' by the 'number of arrangements resulting from each selection'.

Defining ${}^{n}C_{r}$

The number of ways in which r objects can be selected from n distinct objects is denoted by ${}^{n}C_{r}$.

Each selection of r objects can be further arranged among themselves in $r!$ ways and this will result in the total possible ways of arranging r objects from n distinct objects i.e. ${}^{n}P_{r}$.

Thus, ${}^{n}C_{r} \times r! = {}^{n}P_{r} \quad \Rightarrow {}^{n}C_{r} = \dfrac{{}^{n}P_{r}}{r!}$

We already know that ${}^{n}P_{r} = \dfrac{n!}{(n-r)!}$ and thus, ${}^{n}C_{r} = \dfrac{n!}{(n-r)! \times r!}$

Obviously this is just the formula of ${}^{n}C_{r}$, but when we are finding the value of ${}^{n}C_{r}$, we would do the following calculation ....

$${}^{n}C_{r} = \frac{\underbrace{n \times (n-1) \times (n-2) \times \ldots\ldots}_{r\ terms}}{r!} \text{ i.e. } {}^{15}C_{4} = \frac{15 \times 14 \times 13 \times 12}{1 \times 2 \times 3 \times 4} \text{ and } {}^{10}C_{3} = \frac{10 \times 9 \times 8}{1 \times 2 \times 3}$$

A few values that one should be very conversant with are...

${}^{n}C_{1} = n$.

(And this is obviously so since if we have to select 1 out of n distinct objects, we can select the first or the second or the third or ...... or the n^{th} i.e. in n different ways. Please remember that ${}^{n}P_{1}$ was also equal to n. This should also be obvious because when there is just 1 object, arranging l object has no meaning and thus will be same as selecting 1 object)

${}^{n}C_{n} = 1$.

(Again this should be obvious since we have to select n out of n i.e. we have to select all of them and we do not have any choice to make the selection in any other manner)

In management entrance exams the questions asked on selection are relatively easier and stick to the basic format of selecting teams or committees.

E.g. 23: In how many ways can a committee of 7 members be chosen from 10 people?

The required answer is $${}^{10}C_{7} = \frac{10 \times 9 \times 8 \times \cancel{7 \times 6 \times 5 \times 4}}{\cancel{7 \times 6 \times 5 \times 4} \times 3 \times 2 \times 1} = \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 120$$

${}^{n}C_{r} = {}^{n}C_{n-r}$

As seen in the above expression, we had to write a lot of factors both in the numerator and the denominator, only to find that most of them cancel out.

Further if one is very observant, one can realise that after cancelling out the terms, we get a term which is exactly similar to the calculations of ${}^{n}C_{r}$ i.e. $\dfrac{10 \times 9 \times 8}{3 \times 2 \times 1} = {}^{10}C_{3}$

Thus, looking again at the calculation, $${}^{10}C_{7} = \frac{10 \times 9 \times 8 \times \cancel{7 \times 6 \times 5 \times 4}}{\cancel{7 \times 6 \times 5 \times 4} \times 3 \times 2 \times 1} = \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = {}^{10}C_{3}$$

In general ${}^{n}C_{r} = {}^{n}C_{n-r}$

Mathematical explanation:

$${}^{n}C_{n-r} = \frac{n!}{(n-(n-r))! \times (n-r)!} = \frac{n!}{r! \times (n-r)!} = {}^{n}C_{r}$$

Logical explanation:

Consider the case of selecting *r* objects out *n* distinct objects. For every possible distinct selection, there would be a distinct left-over group of $(n - r)$ objects. Thus, the number of ways of selecting *r* objects would also be exactly equal to the number of ways of forming groups of $(n - r)$ objects from *n* objects.

Say we have to select 20 people from a group of 25 people. Rather than calling out the names of the 20 people who are selected, it is far easier to call out the names of 5 people who are not selected. Every possible ways of selecting 20 people has a unique group of 5 people not selected.

The above can be very useful to save our calculation when the number of objects to be selected is more than half the total objects.

E.g. 24: Out of 5 men and 6 women in how many ways can a committee of 2 men and 3 women be selected?

2 men can be selected out of 5 men in $^5C_2$ ways

3 women can be selected out of 6 women in $^6C_3$ ways.

Since we have to select 2 men *and* 3 women, it can be done in $^5C_2 \times {}^6C_3$ =

$$\frac{5\times4}{2!}\times\frac{6\times5\times4}{3!} = 10 \times 20 = 200 \text{ ways.}$$

E.g. 25: Out of 5 men and 6 women in how many ways can a committee of 5 members be selected such that atleast 3 members are women?

Atleast 3 members are women implies that there could be 3 women *or* 4 women *or* 5 women in the committee. Thus, the required number of ways is

$$\underbrace{{}^6C_3 \times {}^5C_2}_{\text{3 women } and \text{ 2 men}} + \underbrace{{}^6C_4 \times {}^5C_1}_{\text{4 women } and \text{ 1 man}} + \underbrace{{}^6C_5}_{\text{5 women}}$$

$$= \frac{6\times5\times4}{3!}\times\frac{5\times4}{2!}+\frac{6\times5\times4\times3}{4!}\times\frac{5}{1!}+\frac{6\times5\times4\times3\times2}{5!}$$

$$= 20 \times 10 + 15 \times 5 + 6 = 281$$

However there are a few questions where you would need to identify the application of Selection and is not very apparent. These types of question could appear to be from any topic like reasoning (as seen in the following example) or geometry (as seen in few questions in the exercise)

E.g. 26: In a room there are 10 men and each man shakes hands with every other man present. How many hand-shakes take place?

A hand shake happens between every pair of person. Thus the number of hand-shakes is equal to the number of different pair of persons that can be formed from 10 people. The number of ways of choosing 2 persons out of 10 is $^{10}C_2 = \frac{10\times9}{2!}$ = 45. Thus 45 hand-shakes take place.

Alternate ways using reasoning and not P&C

Method 1: Each person is going to shake hands with 9 other people. And since there are 10 people, the total number of handshake will be 10 × 9. But then each hand-shake will be counted twice, once from each person's perspective and so the correct number of handshakes will be $\frac{10 \times 9}{2}$

Method 2: The first person will shake hands with 9 other people. The number of handshakes of the second person that are not yet counted are 8; the number of handshakes of the third person which are not yet counted are 7; and so on till the number of handshakes of the ninth person that are not yet counted is 1 and all handshakes of the tenth person would have been already be counted. Thus, total number of handshakes = $9 + 8 + 7 + \ldots\ldots + 1 = \frac{9 \times 10}{2}$. (Sum of first n natural numbers is $\frac{n \times (n+1)}{2}$)

Exercise

37. Each of three schools has sent a team of 4 for the competition. But only two from each school have to be selected. In how many ways can the selection be made?

(1)18 (2) 36 (3) 216 (4) 1278

38. In 16 probable's for the team there are 6 bowlers, 2 wicket-keepers and the rest are all batsmen. In how many ways can a team of 11 be selected such that 1 keeper and atleast 5 bowlers are chosen?

(1) 672 (2) 812 (3) 476 (4) 1288

39. In how many ways can a team of 5 be selected from 8 people such that either all three specific persons from the eight are selected or none of them are selected?

(1) 11 (2) 120 (3) 140 (4) 10

40. In round-robin matches, each team plays every other team once. If there are 8 teams participating in the round-robin league matches, how many matches will be played?

(1) 7 (2) 14 (3) 21 (4) 28

41. In how many ways can a committee of a president, vice-president and a treasurer be selected from 5 persons?

(1) 10 (2) 30 (3) 50 (4) 60

42. How many word containing 2 distinct vowels and one consonant can be made from a given set of 4 distinct vowels and 3 distinct consonants?

(1) 36 (2) 54 (3) 72 (4) 108

43. How many diagonals can be formed in a regular decagon? A decagon is a polygon with 10 vertices.

(1) 35 (2) 45 (3) 70 (4) 90

44. On a plane there are 10 points, no three of which are collinear. How many triangles can be formed such that the vertices of the triangle are these points?

(1) 720 (2) 640 (3) 120 (4) 60

45. In a plane, there is a set of 5 parallel lines and another set of 6 parallel lines, each line of the second set intersecting each line of the first set to form a sort of grid. How many parallelograms are present in this grid?

(1) 25 (2) 50 (3) 75 (4) 150

46. From 14 probables, in which Sachin and Rahul are included, a team of 11 has to be selected. In how many ways can the team be selected such that if Sachin is selected, Rahul should not be selected and if Rahul is selected then Sachin should not be selected?

(1) 132 (2) 78 (3) 286 (4) 572

A tricky error

Consider a questions exactly similar to the example 25 but with larger numbers: “In how many ways can a committee of 7 be selected from 10 women and 11 men such that atleast 4 of the selected members are women?”.

The solution to this is ${}^{10}C_4 \times {}^{11}C_3 + {}^{10}C_5 \times {}^{11}C_2 + {}^{10}C_6 \times {}^{11}C_1 + {}^{10}C_7$.

This is a longish expression. Even working by excluding the committees that are not acceptable (0 women, 1 women, 2 women and 3 women) from the all possible committees would be equally lengthy.

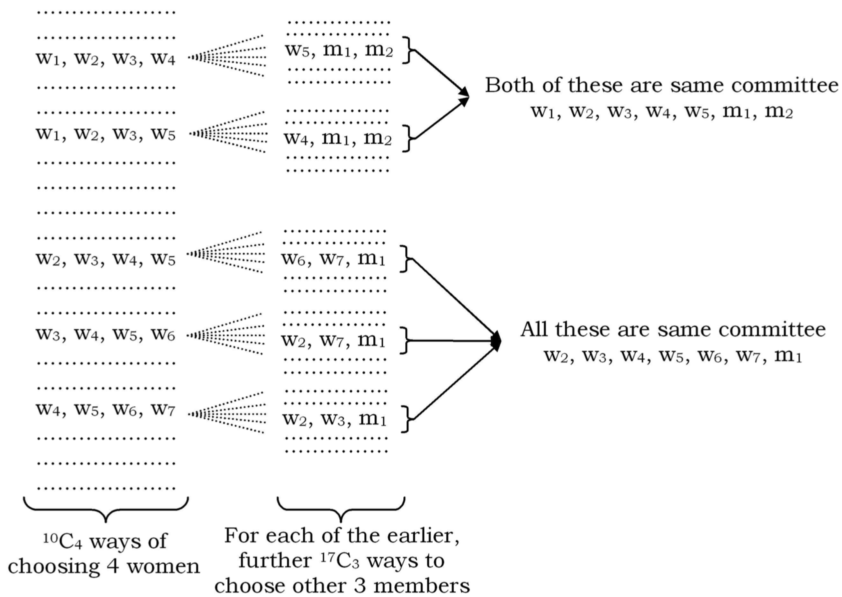
A very tempting way to avoid this longish expression is as follows: Select any 4 women out of the 10 women. This can be done in ${}^{10}C_4$. And now from the remaining 6 women and 11 men i.e. from 17 people, select any 3 persons. Even if all three selected are men, our condition of alteast 4 women being in the committee is already met. If all the three selected are women, it will be the committee of 7 women as included in the earlier solution as well.

Thus the solution arrived at by this logic will be ${}^{10}C_4 \times {}^{17}C_3$, a far shorter expression than before.

However this approach is WRONG. And it would pay well if one learns why this is wrong and how to avoid such errors, which will be tempting in other situations also (see grouping as given in a later topic)

The following figure highlights how one would be counting many committees more than once in the answer found by this approach...(w_1, w_2, w_3, ...refer to the 10 different women and m_1, m_2, ... refer to the 11 different men)

...... cont'd



Thus, in this approach there is a lot of multiple counting. Further different committees will be counted different number of times and hence there is no way to correct it.

Why does this error occur? Because we have considered the committee of 7 as two different groups, first group of 4 women and then a second group of 3 members. Further since we have multiplied ${}^{10}C_4$ with ${}^{17}C_3$, we have also ordered the two groups. But the committee of 7 is a single team and does not consist of an ordered group of 4 and 3.

Another way to look at it is that a woman now has two ways of being selected to the committee, being a part of the mandatory 4 or being one among the three selected from the remaining. And thus a committee will be counted more than once based on which way the women is selected to the committee.

Selecting any number of objects

In certain situations one has the liberty of selecting any number of objects from say n given objects. In this case one can select 1 object or 2 objects or 3 objects or so on or all n objects. Just as one could have selected n objects, one could also have selected none of the given objects i.e. selected 0 objects.

Further, if the n objects are all different objects then not just how many are to be selected but a further question of which objects are selected also assumes importance. Thus there are two cases viz. the n objects being distinct or being identical.

Selecting any number from n distinct objects

We could select 0 or 1 or 2 or 3 or …… or all n.

Say we are selecting 3 objects. Since the n given objects are distinct one, we could select 3 objects in not just 1 way but in ${}^nC_3$ ways.

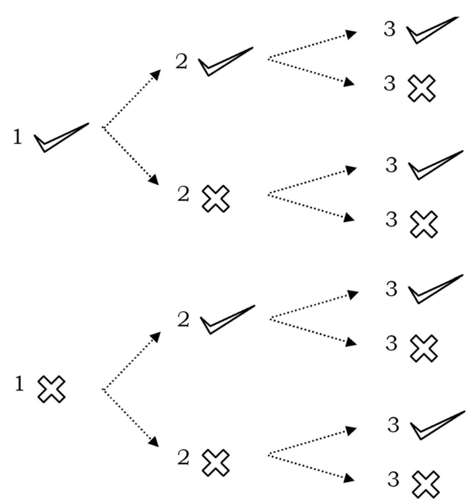
Thus, the total number of ways of selecting any number of objects would be ${}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots\ldots + {}^nC_n$.

Those who are familiar with binomial expansion would recollect that ${}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots\ldots + {}^nC_n = 2^n$. Those not familiar, just take this for granted for the moment. Shortly we shall see a logical explanation for the same without using any binomial expansion.

Thus the number of ways of selecting any number of objects from n distinct objects is 2^n and the number of ways of selecting atleast one object from n distinct objects is $2^n - 1$ (subtract the 1 way of not selecting 0 objects from the total 2^n)

Explanation for 2^n

If we look at each object individually, there are two options with each object viz. either selecting the object or not selecting. Irrespective of what option is exercised with the first object, the second object also has two options. And so on. Look at the following figure, to see how for three objects, the 'tree structure' finally has 8 branches, with each branch resulting in a unique way of selecting the objects.



Tick is for selected and Cross is for not selected.
For every object, the earlier number of ways gets multiplied by 2. Thus, for n objects, the number of ways will be $2 \times 2 \times 2 \times \ldots = 2^n$

E.g. 27: A buffet dinner consists of 5 different dishes. In how many different ways can one help oneself if he has to take atleast one dish?

The person can help himself to 1 or 2 or 3 or 4 or all 5 dishes. Further when he takes 1 (or 2 or 3 or 4) he can also choose which of the dish he takes. Thus he can help himself in ${}^5C_1 + {}^5C_2 + \ldots + {}^5C_5$ i.e. $2^5 - 1 = 31$ ways.

Selecting any number from *n* identical objects

Say one has to select 4 books from a set of 10 books, all of which are the same copy of "My Experiments with Truth". Since all the books are identical, there really is no choice. Irrespective of whichever physical copy one selects, the outcome is the exactly the same: 4 copies of "My Experiments with Truth". Thus, there is only one way in which 4 objects can be selected from 10 identical objects.

Similarly, if one had to select 5 books from the earlier set of 10 identical books, it could again be done in only 1 way.

When we have *n* identical objects and we can select any number of them, in what way can a selection differ from another selection?

A selection can be distinct from another selection only when the number of objects selected is different. Thus, from *n* identical objects, one could have selected 0 or 1 or 2 or 3 or … or all *n*. And each of these ways could be performed in exactly 1 way (Since all the objects are identical, the question of which object does not arise). Thus the total number of ways would be:

Number of objects selected:	0	or	1	or	2	or	3	or	……	or	n	
Number of ways:	1	+	1	+	1	+	1	+	……	+	1	$= (n + 1)$

In maths, fruits and flowers of a kind are all identical unless the question specifies it differently. Thus from a pile of 8 apples, one can select any number of apples in 9 different ways and if one had to select atleast one apple, the number of ways in which the selection can be made is 8.

E.g. 28: In how many different ways can a person make a purchase from a fruit seller who has 5 mangoes, 8 apples and 10 oranges left with him and if the person has to purchase atleast 1 mango, atleast 1 apple and atleast 1 orange?

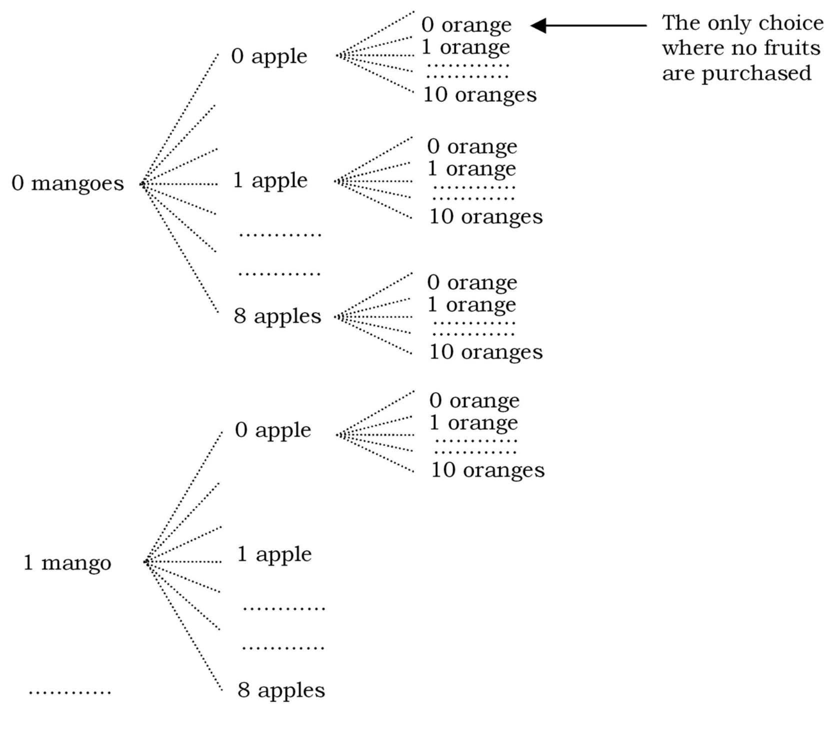
Since atleast 1 of each type has to be purchased, the number of ways with each of the different fruits is 5 ways, 8 ways and 10 ways. Thus, the total number of ways in which the purchase can be made is $5 \times 8 \times 10 = 400$ ways.

E.g. 29: In how many different ways can a person make a purchase from a fruit seller who has 5 mangoes, 8 apples and 10 oranges left with him and if the person has to purchase atleast 1 fruit?

Looking independently at the number of mangoes, apples and oranges bought, it is quite possible that one purchases 0 mangoes; purchasing 0 apples is also possible; and similarly it is quite possible that in some purchase 0 oranges are purchased (some other fruit, maybe an apple and/or a mango is purchased). Thus looking at each of the type of fruit individually, it is quite possible to purchase 0 of them (BUT zero of each of the three variety is not possible).

Thus with mangoes, the person has 6 choices (purchasing 0 or 1 or 2 or 3 or 4 or 5 mangoes). Similarly, with apples, the person has 8 + 1 = 9 ways to make a purchase and with oranges the person has 10 + 1 = 11 ways.

Thus in all there are 6 × 9 × 11 = 594 ways but one of these ways is where the person has bought 0 mango, 0 apple and 0 orange. Subtracting this case, the total number of ways of making a purchase of atleast one fruit is (5 + 1) × (8 + 1) × (10 + 1) – 1 = 593. See figure below for more clearer understanding...



Grouping

Consider that 12 people have to be divided among three groups such that one group has 3 members, one group has 4 members and one group has 5 members?

We could have formed the group of 3 members in $^{12}C_3$ ways. Having formed a group of three, we would be left with 12 – 3 = 9 people. A group of 4 members can be formed from these 9 members in $^{9}C_4$ ways. For each group of 3 members formed earlier, there would be further $^{9}C_4$ ways of forming the group of four. Thus, the total possible number of ways of forming a group of 3 and a group of 4 would be $^{12}C_3 \times {}^{9}C_4$. Now there would be 5 people left who are the third group i.e. the third group can be formed in only 1 way. To maintain consistency, we will say that the third group can be formed in $^{5}C_5$ ways (which is 1 anyways). Thus total number of ways of forming the groups is $^{12}C_3 \times {}^{9}C_4 \times {}^{5}C_5$

What if we had formed the group of 4 members first, then the group of 5 members and then the group of 3 members? In this case the number of ways would have been $^{12}C_5 \times {}^{7}C_4 \times {}^{3}C_3$.

A little working would show that either these expressions (or the other similar expressions that can be formed by changing the order in which groups are formed) will boil down to the same value...

$$^{12}C_3 \times {}^{9}C_4 \times {}^{5}C_5 = \frac{12\times11\times10}{3!}\times\frac{9\times8\times7\times6}{4!}\times\frac{5\times4\times3\times2\times1}{5!} = \frac{12!}{3!\times4!\times5!}$$

$$^{12}C_5 \times {}^{7}C_4 \times {}^{3}C_3 = \frac{12\times11\times10\times9\times8}{5!}\times\frac{7\times6\times5\times4}{4!}\times\frac{3\times2\times1}{3!} = \frac{12!}{3!\times4!\times5!}$$

Thus, the number of ways of dividing 12 distinct objects into groups of 5, 4 and 3 is $\frac{12!}{3!\times4!\times5!}$

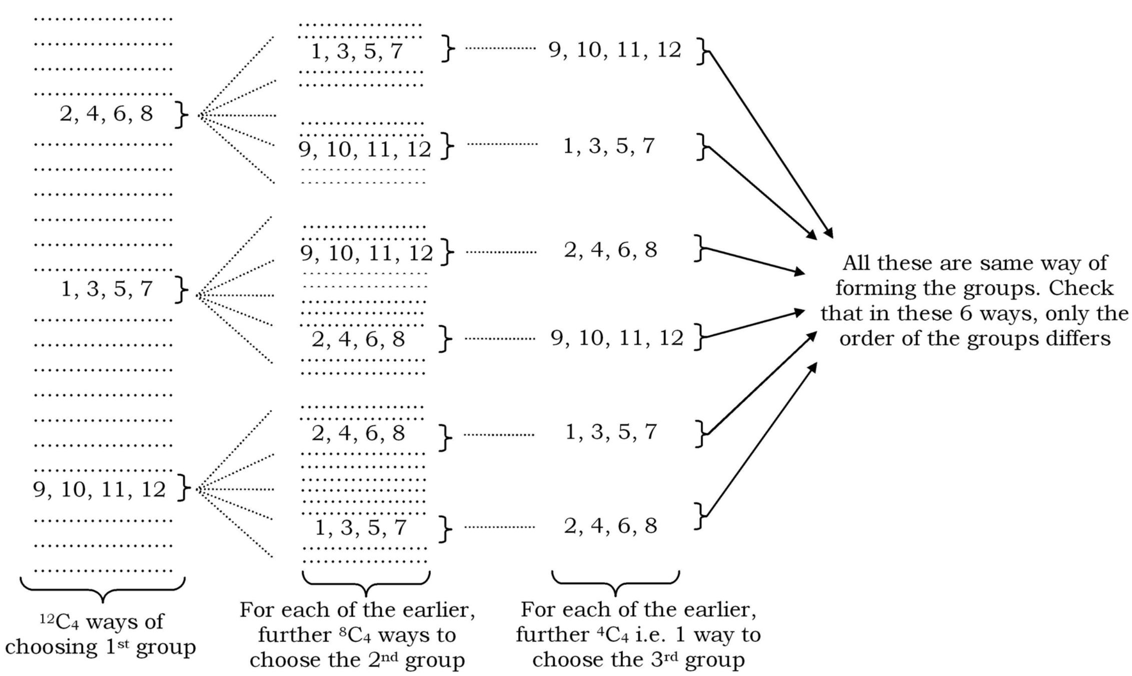
When two or more groups are of the same size

By the same logic, the number of ways of dividing 12 distinct objects into 3 equal groups (each of 4 members) would be $^{12}C_4 \times {}^{8}C_4 \times {}^{4}C_4$ which will boil down to $\frac{12!}{4!\times4!\times4!}$. This answer is wrong, for this case, when two or more group have the same size, and has to be further corrected.

The number of ways of forming the three groups is $^{12}C_4$, $^{8}C_4$, $^{4}C_4$. Since we are multiplying these three factors, we are inadvertently also arranging the groups in a particular order. (Remember that if one position can be filled in 5 ways, another can be filled in 4 ways, third can be filled in 4 ways, when we apply the rule of *AND* i.e. 5 × 4 × 3, we are basically finding the number of "arrangements" of the three positions)

But the question requires us to just form groups and we do not have to "arrange" the groups. Since we have arranged 3 objects which did not have to be arranged, we have counted each unique way of forming the groups 3! times i.e. 6 times. Thus, the correct answer would be found by dividing the earlier found answer by 3!

Look at the following figure for a clearer understanding of how the groups have been arranged ...



Ordered Groups

If we had to divide the 12 people among three groups of 4 each and if the groups had been named differently, say the Scorpio group, the Libra group and the Capri group the answer would have been just $\frac{12!}{4!\times 4!\times 4!}$, the reason being that naming of the groups is same as ordering the groups.

E.g. 30: Find the number of ways of dividing 12 people among three groups such that one group has 6 members and each of the other two groups has 3 members.

It would be $\frac{12!}{6!\times 3!\times 3!}\times\frac{1}{2!}$. In this case we are dividing by 2! because the group of 6 is ordered because of its size, but the two identical groups of 3 members each do not have to be ordered.

Ordered Groups and Un-ordered Groups – Error Prone Area

When groups have the same size, even if the question requires you to find the number of "ordered" groups (when groups are given different names or hierarchy), it is strongly suggested that you first find the number of un-ordered groups that can be formed and now do the required ordering or arranging of the so formed groups with the given names or hierarchy.

E.g.: A group of 10 students are to be divided into three groups, one group of 4 and two groups of 3 each and further one group has to be sent to each of Mumbai, Pune and Bangalore for a research project. In how many different ways can be the whole exercise be done?

If a student assumes that since the groups are ordered (because of them being sent to three different locations), the answer would just be $\frac{10!}{4!\times 3!\times 3!}$ and no correction is needed, one would go wrong.

The suggested way is to first form un-ordered groups. This can be done in $\frac{10!}{4!\times 3!\times 3!}\times\frac{1}{2!}$ ways. The correction factor of dividing by 2! is because two groups are of the same size and hence would have got arranged, whereas we are forming un-ordered groups.

Now we have three distinct groups and they have to be sent to three different locations and this can be done in 3! ways.

Since this 3! ways of arranging the groups to the three locations can be done for each of the ways of forming the un-ordered groups, the total possible number of ways in which the exercise can be done is $\frac{10!}{4!\times 3!\times 3!}\times\frac{1}{2!}\times 3! = \frac{10!}{4!\times 3!\times 3!}\times 3$.

Exercise

47. A florist has 8 red carnations and 5 roses, each of a different colour. In how many different ways can I purchase atleast 1 flower?

(1) 287 (2) 279 (3) 248 (4) 256

48. Find the number of different sums that can be formed from one 25 ps coin, one 50 ps coin, one 1 Re coin, one 2 Rs coin and one 5 Rs. Coin.

(1) 19 (2) 31 (3) 35 (4) 50

49. In a city no two persons have identical set of teeth and there is no person without a tooth. Also no person has more than 32 teeth. If we disregard the shape and size of tooth and consider only the positioning of teeth, then find the maximum population of the city.

(1) $32^2 - 1$ (2) 32^2 (3) 2^{32} (4) $2^{32} - 1$

50. If in the English language there were only 6 alphabets and repetition of alphabets in the same word were not allowed, how many distinct possible words could possibly exist?

(1) 6! (2) 6^6 (3) $1.2 \times (6^6 - 1)$ (4) $2^6 - 1$

51. In how many ways can a pack of 52 playing cards be divided into 4 sets such that 3 of the sets have 17 cards and the fourth has just 1 card.

(1) $\frac{52!}{(17!)^3}$ (2) $\frac{52!}{(17!)^3 \times 3!}$ (3) $\frac{52!}{(13!)^4 \times 4!}$ (4) $\frac{52!}{(3!)^{17}}$

52. In how many ways can 10 men accommodate themselves in 5 distinct tents such that each tent has 2 people occupying them?

(1) $\frac{10!}{2^5}$ (2) $\frac{10!}{2^5 \times 5!}$ (3) $\frac{10!}{2^5 \times 5}$ (4) $\frac{10!}{2^5} \times 5!$

53. The number of ways of dividing $2n$ people into n couples is.......

(1) $\frac{(2n)!}{(n!)^2 \times 2!}$ (2) $\frac{(2n)!}{(n!)^2 \times n!}$ (3) $\frac{(2n)!}{(2!)^n \times n!}$ (4) $\frac{(2n)!}{(2!)^n \times 2!}$

54. There are n different books and p copies of each in a library. The number of ways in which one or more than one can be selected is

(1) $p^n - 1$ (2) $n^p - 1$ (3) $(n + 1)^p - 1$ (4) $(p + 1)^n - 1$

Distribution

While arranging 8 people in eight chairs is also a question similar to distributing people among chairs. However this topic is characterised by two distinct features:

Usually the number of objects to be distributed are far more than the number of people/boxes. E.g. Distribute 8 chocolates among three children.

And MORE IMPORTANTLY each box/person can receive more than one object (unlike making people sit in a row or forming a number, where each position can be taken up by only one object)

Distributing Distinct objects

In how many ways can 8 different chocolates be distributed among three children such that each can get any number of chocolates (including zero)?

The first chocolate can be given in 3 ways. The second chocolate can also be given in 3 ways (can also be given to the same child who got the first chocolate). Similarly the third chocolate can also be given in 3 ways and so on for all the 8 chocolates. Thus the answer is $3 \times 3 \times 3 \times \ldots \times 3 = 3^8$.

3^8 or 8^3

Please remember that even the most intelligent person does make this silly error and so the suggestion is to speak out the entire above sentences (first chocolate in 3 ways, second in 3 ways ... and hence $3 \times 3 \times$ ...) in your mind before committing to the answer.

The above reasoning is with the perspective of what happens to each chocolate i.e. first chocolate can be given in 3 ways. Few students do have a doubt about why don't we solve the problem from the perspective of a child i.e. the first child can receive chocolate in xx ways, second in yy ways and so on. Read on to see how difficult this approach is ...

In how many ways can the first child receive chocolates?

If your answer is 8 ways, think again. If your reasoning is that he could get 1 chocolate or 2 chocolate or ...... or all 8 chocolates, could he not get zero chocolates? So should the answer be 9 ways? Not really, remember that the chocolates are different, so the child getting 1 chocolate is not just 1 way and which chocolate he gets also matters. Well, this can also be taken care of. This is a case of the first child getting any number of chocolates out of 8 distinct chocolates. Thus, he can get receive chocolates in 2^8 ways.

The hurdle in this approach starts after this: after the first child having received chocolates, in how many ways can the second child receive chocolates? The answer to this depends on what the first child has got....

If the first child gets 0 chocolates, the second child can get chocolates in 2^8 ways

If the first child gets 1 chocolate, the second child can get chocolates in 2^7 ways

And so on, such that if the first child gets 7 chocolates, the second child can get chocolates in just 2 ways (gets the remaining chocolate or gets no chocolate)

And lastly, if the first child gets 8 chocolates, the second child can get chocolates in only 1 way i.e. no chocolate.

Thus, we would have to use a case of *or* because the number of ways of the second child receiving chocolates is different for each case of the first receiving chocolates (each case has to be handled independently). The problem gets compounded for the third child as the number of ways he gets chocolates is dependent on both of what the first gets and what the second gets.

E.g. 31: In how many ways can 8 letters be posted in 8 letter boxes?

The first letter can be posted in 8 ways, second in 8 ways and so on. Thus the answer is a straight-forward 8^8 ways.

> When is the answer 8! and when is it 8^8?
>
> When 8 people have to arranged in 8 chairs the answer is 8!. In this case each chair can accommodate only one person.
>
> When 8 letters have to be posted in 8 letter boxes, the answer is 8^8. In this case each letter box can accommodate any number of letters.

E.g. 32: In how many ways can 6 distinct hats be distributed in 3 distinct boxes such that each box receives atleast 1 hat?

This case of each box receiving atleast 1 hat is the most confusing to students. Unfortunately the answer is not so straight as 3^6 or anything similar to this.

Step 1: We would have to individually consider all possible cases of different number of hats in the boxes.

In this question, the boxes (in no particular order) could contain hats numbering {1, 1, 4} or {1, 2, 3} or {2, 2, 2}

Step 2: For each of the above case, divide the 6 hats into three "ordered" groups of sizes as in each case. We need to form "ordered groups" because the boxes are distinct.

Case 1: Divide 6 hats in three groups with sizes 1, 1, and 4: Using the logic learnt in grouping, "unordered" groups can be formed in $\frac{6!}{1!\times1!\times4!}\times\frac{1}{2!}$

These three groups can now be arranged in the three distinct boxes in 3! ways and hence the total number of ways of distributing hats for this case is $\frac{6!}{1!\times1!\times4!}\times\frac{1}{2!}\times3!$ = 90 ways.

Case 2: Divide 6 hats in three groups with sizes 1, 2, and 3: Using the logic learnt in grouping, since the size of all groups is different, the number of groups can be formed in $\frac{6!}{1!\times2!\times3!}=60$ ways. These three groups can now be arranged in the three distinct boxes in 3! ways and hence the total number of ways of distributing hats for this case is 60 × 3! = 360 ways.

Case 3: Divide 6 hats in three groups with sizes 2, 2, and 2: Using the logic learnt in grouping, "unordered" groups can be formed in $\frac{6!}{2!\times 2!\times 2!}\times\frac{1}{3!}$

These three groups can now be arranged in the three distinct boxes in 3! ways and hence the total number of ways of distributing hats for this case is $\frac{6!}{2!\times 2!\times 2!}\times\frac{1}{3!}\times 3! = 90$ ways.

Thus, the final answer would be 90 + 360 + 90 = 540 ways.

The case when boxes are also identical

The question could be: In how many ways can 6 distinct hats be distributed in 3 identical boxes?

This question in essence is just about forming groups. The hurdle is that the group sizes is not defined and could be anything. Thus we would have to solve this question in a similar manner as the above solved example i.e.

Step 1: We would have to individually consider all possible cases of different number of hats in the boxes.

Step 2: For each of the above case, divide the 6 hats into three "un-ordered" groups of sizes as in each case. Thus, the only difference is that we are forming "un-ordered" groups. So the last step of 'multiplying by 3!' in each of the case in above example will not have to be done.

Case 1: Distribute into groups of {0, 0, 6}: Can be done in only 1 way

Case 2: Distribute in groups of {0, 1, 5}: Can be done in $\frac{6!}{0!\times 1!\times 5!} = 6$ ways

Case 3: Distribute in groups of {0, 2, 4}: Can be done in $\frac{6!}{0!\times 2!\times 4!} = 15$ ways

Case 4: Distribute in groups of {0, 3, 3}: Can be done in $\frac{6!}{0!\times 3!\times 3!}\times\frac{1}{2!} = 10$ ways

Case 5: Distribute in groups of {1, 1, 4}: Can be done in $\frac{6!}{1!\times 1!\times 4!}\times\frac{1}{2!} = 15$ ways

Case 6: Distribute in groups of {1, 2, 3}: Can be done in $\frac{6!}{1!\times 2!\times 3!} = 60$ ways

Case 7: Distribute in groups of {2, 2, 2}: Can be done in $\frac{6!}{2!\times 2!\times 2!}\times\frac{1}{3!} = 15$ ways

Thus, total possible ways is 1 + 6 + 15 + 10 + 15 + 60 + 15 = 122 ways

If the question had the condition that each box has to receive atleast 1 hat, then we would only consider cases 5, 6 and 7.

Distributing Identical Objects

In how many ways can 8 identical chocolates be distributed among three children such that each can get any number of chocolates (including zero)?

Why is this different from distributing different chocolates?

In this case also we could reason as: the first chocolate can be distributed in 3 ways, the second in 3 ways, the third in 3 ways and so on. Thus the total number of ways of distributing the 8 chocolates would be $3 \times 3 \times 3 \times \ldots\ldots \times 3 = 3^8$

While the above approach would seem convincing enough in this case as well, it is not so. When the chocolates are distinct (say C_1, C_2, C_3, ...), the following two ways of distributing the chocolates is distinct...

	Child 1	Child 2	Child 3
Way 1 of distributing:	C_1 C_2	C_3 C_4 C_5	C_6 C_7 C_8
Way 2 of distributing:	C_1 C_3	C_5 C_7 C_8	C_2 C_4 C_6

But if the chocolates are identical, i.e. if there had been no subscript attached to the C, both the above way would have been the same.

When the chocolates are identical, different ways of distributing the chocolates are not because of 'which' chocolates one gets but ONLY because of 'how many' chocolates one gets.

Since different ways of distributing the chocolates is just dependent on the number of chocolates that each gets (and not about which chocolate), let the first child get a chocolates, second child gets b chocolates and third child get c chocolates. The condition on the variables is that they have to be whole numbers and $a + b + c = 8$.

Each whole number solution to $a + b + c = 8$ would result in a different way of distributing the chocolates. And we have already learnt that the number of whole number solution to the equation is same as the number of ways in which eight one's and two partitions i.e. 11111111*pp* can be arranged. Thus the answer is $\frac{10!}{2! \times 8!} = 55$ ways.

E.g. 33: In how many ways can 12 identical hats be put in 3 boxes such that each box has atleast 2 hats?

Since the hats are identical, different ways of distributing would depend on the number of hats in the boxes. If a, b, and c refer to the number of hats in the boxes, then the answer is same as the number of whole number solutions to $a + b + c = 12$ with the condition that each of a, b, and c has to be atleast 2. Giving 2 hats to each of the box, we are left with 6 hats and thus, we need whole number solutions to $a + b + c = 6$, with no conditions on a, b, and c. Thus the answer is $\frac{8!}{2! \times 6!} = 28$

The case when boxes are also identical

Had the question in above example been: In how many ways can 12 identical hats be put in 3 identical boxes such that each box has atleast 2 hats?

First and foremost 2 hats should be put in each box (which two does not matter as all are identical)

Now we are left with 6 hats to be put in 3 identical boxes. As learnt earlier, in the case of identical boxes, we are only concerned with grouping the 6 hats in 3 "un-ordered" groups. The hurdle is that the group sizes could be anything.

Thus again we individually consider all case of grouping 6 hats into 3 groups, with groups being every possible size. The possibilities are {0, 0, 6}; {0, 1, 5}; {0, 2, 4}; {0, 3, 3}; {1, 1, 4}; {1, 2, 3}; {2, 2, 2} i.e. 7 possibilities.

The fortunate part is that we do not have to do anything further and 7 is our answer because each possible way of grouping can be done in only 1 way. Why? Because all hats are identical, so "which hat is in which group" does not matter.

Thus, answer is 7 ways.

Exercise

55. In how many ways can five distinct antiques be divided among two brothers, such that each gets atleast one antique piece?

(1) $2^5 - 2$ (2) 2×2^3 (3) $^4C_1$ (4) $^5C_2 + {^5C_3} + {^5C_4}$

56. A father has five distinct antiques and he is thinking of giving them to his two sons. In how many ways can he do so if it is possible that he may not necessarily give away all the 5 antiques. Also a son may or may not receive an antique.

(1) $2 + 2^2 + 2^3 + 2^4 + 2^5$ (2) $2^6 - 1$ (3) 3^5 (4) $3^5/2$

56. In how many ways can 5 scholarships, each having different value, be distributed among 20 students if a student can receive any number of scholarships?

(1) 20^5 (2) 5^{20} (3) $^{24}C_5$ (4) $^{20}C_5$

57. In how many ways can 5 scholarships, each having different value, be distributed among 20 students if a student cannot receive more than one scholarship?

(1) $^{20}C_5$ (2) $^{20}P_5$ (3) $^{24}C_5$ (4) $^{24}P_5$

58. In how many ways can 5 scholarships, each of the same value, be distributed among 20 students if a student cannot receive more than one scholarship?

(1) $^{20}C_5$ (2) $^{20}P_5$ (3) $^{24}C_5$ (4) $^{24}P_5$

59. In how many ways can 20 scholarships, each of same value, be distributed among 5 students such that each student gets atleast one scholarship?

(1) $^{24}C_4$ (2) $^{24}C_5$ (3) $^{19}C_4$ (4) $^{19}C_5$

60. In how many ways can 5 scholarships, each of the same value, be distributed among 20 students if a student can receive any number of scholarships?

(1) $^{20}C_5$ (2) $^{20}P_5$ (3) $^{24}C_5$ (4) $^{24}P_5$

61. In how many ways can 6 distinct hats be put in 4 distinct boxes such that no box is empty?

(1) 4095 (2) 1295 (3) 620 (4) 2640

62. In the game 'Ring in a Peg', a child has to place 5 distinctly coloured rings in three pegs placed side by side. In how many distinct ways can the child put the rings in the pegs?

(1) 3^5 (2) 5^3 (3) $\frac{7!}{2!}$ (4) $\frac{7!}{5!}$

62. In a zoo there are 8 cages arranged in a row and numbered I to VIII. The zoo has lions, tigers and leopards, not less than 8 of each. In how many ways can 8 of these animals be assigned to the 8 cages, each cage having one animal?

(1) 8! (2) 8^8 (3) 3^8 (4) 8^3

Appendix

We have already studied the formulae for ^nP_r and ^nC_r as follows:

$$^nP_r = \frac{n!}{(n-r)!} = \underbrace{n \times (n-1) \times (n-2) \times ......}_{r \text{ factors}}$$

$$^nC_r = \frac{n!}{(n-r)! \times r!} = \frac{\overbrace{n \times (n-1) \times (n-2) \times ......}^{}\!\!\underbrace{}_{r \text{ factors}}}{r!}$$

There can be many questions based on the use of these formula, where a variable is used. Though these types of questions have never appeared in CAT, the following questions are provided here, so that you get comfortable in using these formulas when a variable is in picture. The options are not given because it is very easy to plug in the options and identify the answer, whereas the idea is to work with the above definitions and get acquainted with manipulating factorials.

1. If $^nP_4 = 12 \times {}^nP_2$, find the value of n.

2. If $^nP_4 : {}^{n+1}P_4$ is 3 : 4, find the value of n.

3. If $^{k+4}P_{k-1} = 3 \times {}^{k+3}P_k$, find the value of k.

4. If $^{20}P_r = 6840$, find the value of r.

5. If $^nC_{30} = {}^nC_4$, find the value of n.

6. If $^{18}C_r = {}^{18}C_{r+2}$, find the value of $^rC_6$.

7. If $^nC_{n-4} = 15$, find the value of n.

8. If $^{15}C_r : {}^{15}C_{r-1}$ is 11 : 5, find the value of r.

Answers are given in the Answer Key on last page of the book.

Probability

Probability is the theory of quantifying the chance of an event occurring (or not occurring). While the theory is vast in itself, for the purpose of exam, very elementary questions are asked and hence we will stick to just the basics. All the questions of probability that are asked in entrance exams are based on discrete events and in the case of discrete events, probability is defined as:

$$\text{Probability of an event occurring} = \frac{\text{number of favourable outcomes}}{\text{total number of possible outcomes}}$$

By the formula it should be evident that probability can never be more than 1 because the number of favourable outcomes can never be more than the total number of outcomes.

Examples:

In a case of a coin being tossed, the total number of outcomes is just two viz. heads or tails turning up. If we want to find the probability of heads turning up on a toss, it is simply 1/2.

When a regular dice is rolled, the total number of outcomes is 6 and thus the probability that a prime number is rolled will be greater than 2 will be 4/6 since the favourable number of cases are rolling of 3 or 4 or 5 or 6.

When two cards are drawn out of a pack of cards, the total number of outcomes possible is ${}^{52}C_2$. If we want to find the probability of both the cards being face card, the two cards drawn must be from the 12 face cards and this can happen in ${}^{12}C_2$ ways. Thus the required probability is $\frac{{}^{12}C_2}{{}^{52}C_2}$

Tossing a coin, rolling a dice and drawing cards from a pack of cards are the most common scenarios in questions on probability.

E.g. 1: What is the probability that the sum of numbers turned, when two dice are rolled, is 10?

When two dice is rolled, there a total of 36 outcomes as shown below:

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

Please note that first dice turning 1 and second turning 2 is different from first dice turning 2 and second turning 1.

Required number of cases where the sum is 10 is (6, 4), (5, 5), (4, 6).

Thus the required probability is 3/36.

E.g. 2: One card is drawn out from a pack of cards. What is the probability that the card drawn out is a King or a Red card?

One card can be drawn out in 52 ways.

While there are 26 Red cards and 4 Kings, the number of ways of drawing a King or a red card is not 26 + 4 i.e. 40. This is because 2 of the Kings are already counted in the Red card. Thus the number of cards from which a card favourable to the outcome can be drawn is 28. So the required probability = 28/52 i.e. 7/13.

Independent Events

Two events are said to be independent when the outcome of one of the event does not affect the outcome of the other event.

For two *independent* events, A and B, the probability that event A and B occurs is the product of the probabilities of event A and of event B i.e.

$p(A \text{ and } B) = p(A) \times p(B)$

For examples if a coin is tossed 5 times. The outcome on the second toss is not dependent on the outcome of the first toss. For that matter the outcome on any particular toss is not dependent on the outcome of the previous tosses. Thus each toss is an independent event. To find the probability of all five toss turning up Heads, we just need to multiply the probability of turning a head in each toss five times, i.e.

$$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{32}$$

Probability of a Union (Event *A or* Event *B*)

Let's say the probability of A solving a problem is 2/3 and the probability of B solving the problem is 1/2. What is the probability that the problem is solved?

If we simply add the two probabilities, because we are finding the probability that A *or* B solves the problem, our answer will be $\frac{2}{3} + \frac{1}{2} = \frac{7}{6}$ which is greater than 1 and hence it is not possible.

Our answer is greater than 1 because *A* solving the problem is not exclusive to *B* solving the problem i.e. when we consider *A* solving the problem, we have made no mention of *B* and it could be possible that *B* has also solved the problem, which we independently add again. So the answer is greater than 1.

The way out is,

$p(A\ or\ B) = p(A) + p(B) - p(A\ and\ B)$

If they are independent events, we have already learnt that $p(A\ and\ B) = p(A) \times p(B)$.

In this case, since the two guys solve problems independently, the required probability of the problem being solved is:

$p(A\ or\ B \text{ solving}) = p(A \text{ solving}) + p(B \text{ solving}) - p(A \text{ solving}) \times p(B \text{ solving})$

$$= \frac{2}{3}+\frac{1}{2}-\frac{2}{3}\times\frac{1}{2}=\frac{4+3-2}{6}=\frac{5}{6}$$

The above question could also be solved by breaking the case of the problem being solved into cases that are exclusive to each other as follows:

Problem could be solved in either of the following *exclusive* ways:

A solved and B did not solve

A did not solve and B solved

Both of them solved.

Since these cases are exclusive the probabilities could directly be added and the answer can be found as $\frac{2}{3}\times\frac{1}{2}+\frac{1}{3}\times\frac{1}{2}+\frac{2}{3}\times\frac{1}{2}=\frac{2+1+2}{6}=\frac{5}{6}$

Complement of an Event:

If event A is a favourable event, the complement of A is defined as event A *not* occurring and the complement is denoted as A'.

Now it should be obvious that $p(A) + p(A') = 1$ because a event occurring and it not occurring would encompass all the possible outcomes.

This property of the probabilities of an event and its complement adding up to 1 can be used very effectively in certain cases as follows...

The case of 'Atleast 1'

When we are finding the probability of atleast one of the attempts turning a success, it is best solved by finding the probability of its complement and then subtracting it from 1 to find the answer. Because as just seen $p(A) = 1 - p(A')$.

The complement of 'atleast one attempt being a success' is 'none of the attempts being a success'

E.g. 3: A coin is tossed 6 times. What is the probability that atleast one toss results in a head?

One should immediately realize that this is a case of '*atleast 1*' and in breaking up this event into smaller exclusive events would be labourious as we could have 1 head or 2 head or 3 head or …so on.

The best strategy is to find the probability of the complement: 'probability that no toss results into a head'. Thus all toss should result into tail and

p(all 6 toss resulting in tails) = $\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{64}$

p(atleast one toss being head) = 1 – p(all 6 toss resulting in tails) = $1-\frac{1}{64}=\frac{63}{64}$

E.g. 4: In an oil-field, Reliance, Cairns and Shell are exploring for oil. The probability that Reliance strikes oil is 1/3, that Cairns strikes oil is 1/4 and that Shell strikes oil is 1/5. Find the probability oil is found in the oil-field. Consider each company striking oil independent of each other.

Oil will be found if any of the company strikes oil i.e. atleast one of the company strikes oil. One can solve this by using set theory or by considering exclusive events. While there is a better method to solve this, first we shall solve this by both these methods just to reiterate the methods.

Using set theory, we want to find $p(R\cup C\cup S)$ and we can find this by using the formula,

$$p(R\cup C\cup S)=p(R)+p(C)+p(S)-\big(p(R\cap C)+p(C\cap S)+p(R\cap S)\big)+p(R\cap C\cap S)$$

$$=\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\left(\frac{1}{3}\times\frac{1}{4}+\frac{1}{4}\times\frac{1}{5}+\frac{1}{3}\times\frac{1}{5}\right)+\frac{1}{3}\times\frac{1}{4}\times\frac{1}{5}$$

$$=\frac{20+15+12-(5+3+4)+1}{3\times4\times5}=\frac{36}{3\times4\times5}=\frac{3}{5}$$

A slightly better way is to consider exclusive events…

Any of the following exclusive ways could be possible …

$R\,\not{C}\,\not{S}\quad \not{R}\,C\,\not{S}\quad \not{R}\,\not{C}\,S\quad R\,C\,\not{S}\quad R\,\not{C}\,S\quad \not{R}\,C\,S\quad R\,C\,S$

And the required probability is

$$\frac{1}{3}\times\frac{3}{4}\times\frac{4}{5}+\frac{2}{3}\times\frac{1}{4}\times\frac{4}{5}+\frac{2}{3}\times\frac{3}{4}\times\frac{1}{5}+\frac{1}{3}\times\frac{1}{4}\times\frac{4}{5}+\frac{1}{3}\times\frac{3}{4}\times\frac{1}{5}+\frac{2}{3}\times\frac{1}{4}\times\frac{1}{5}+\frac{1}{3}\times\frac{1}{4}\times\frac{1}{5}$$

$$=\frac{12+8+6+4+3+2+1}{3\times4\times5}=\frac{36}{3\times4\times5}=\frac{3}{5}$$

The above method of considering exclusive cases and then adding probabilities of them is a very useful technique.

However, for this question, the best way is to realize that the required probability involves "atleast 1" and thus finding the probability that none of the company strikes oil. This can occur in only one way that is when none of the company strikes oil. This is $\frac{2}{3}\times\frac{3}{4}\times\frac{4}{5}=\frac{2}{5}$. And the probability that atleast one company strikes oil is $1-\frac{2}{5}=\frac{3}{5}$.

Exercise

1. In a simultaneous throw of two dice, what is the probability of getting a sum greater than 7?

   (1) 1/2 (2) 1/3 (3) 5/12 (4) 7/12

2. A bag contains 6 white and 4 black hats. Two hats are drawn simultaneously at random. What is the probability that both are of the same colour?

   (1) 1/4 (2) 7/15 (3) 1/3 (4) 1/2

3. From a pack of 52 cards, two cards are drawn simultaneously. What is the probability that the cards are of the same suit?

   (1) 4/17 (2) 3/17 (3) 2/17 (4) 1/17

4. In a class there are 15 boys and 10 girls. Three students are selected at random. What is the probability that 2 girls and 1 boy is selected?

   (1) 1/4 (2) 3/29 (3) 1/8 (4) 9/88

5. A box contains 20 bulbs, 12 are defective. Four bulbs are selected at random. What is the probability that atleast one of the four selected is defective?

   (1) 33/323 (2) 1/9690 (3) 14/969 (4) 955/969

6. The probability that A solves the problem is 2/3 and that B solves the problem is 3/4. What is the probability that exactly one of them solves the problem?

   (1) 17/12 (2) 1/2 (3) 5/12 (4) 11/12

7. Group A has 2 boys and 3 girls, group B has 3 boys and 4 girls and group C has 2 boys and 4 girls. One student is selected from each of the group. Find the probability that one girl and two boys are among the three selected.

   (1) 3/4 (2) 1/18 (3) 29/105 (4) 29/315

8. The probability of A, B and C solving a problem is $\frac{2}{3},\frac{2}{5},\frac{3}{8}$. If it is known that exactly one of them solves the problem but it is not known who solves and they receive a prize money of Rs. 600 for solving the problem, in what ratio should they divide this amount among themselves?

   (1) 30 : 10 : 9 (2) 80 : 48 : 45 (3) 9 : 15 : 16 (4) 1 : 1 : 1

Conditional Probability

Conditional Probability is a situation where part information of the outcome changes the probability of an event. Consider the most common example of conditional probability...

Three machines A, B and C produce bolts some of which are faulty. A, B and C respectively produce 100, 200 and 700 bolts and of which 10, 15 and 25 are defective. A quality inspector picks a bolt at random. Find the probability that

a. Find the probability that the bolt picked up is produced by machine A

b. If the bolt picked is defective one, find the probability that the bolt picked up is produced by machine A

Now, the questions ask the same probability "that the bolt picked is produced by machine A", But then in part b, some information of the outcome, namely, the bolt picked is defective, is known.

a. The required probability is a straightforward one $\frac{100}{1000} = \frac{1}{10}$.

b. Since the bolt picked is a defective one, now the sample space is not the 1000 bolts, but is now limited to only the 50 defective ones. And thus, we should now be concerned only with this 'narrow' sample space of 50 bolts. And the required probability is $\frac{10}{50} = \frac{1}{5}$.

If it seems to difficult for you to digest, why should a bolt being defective or not affect the probability of it being from machine A, especially since the event is picking a bolt randomly, think of the following....

Let's say only machine A produced defectives. Now on picking one bolt and finding it to be defective, wouldn't you be absolutely sure that it is from machine A.

Thus, knowing part of the information of the outcome reduces the sample space.

Consider one more example...

There are two urns, one having 5 white marbles and 6 black marbles and the other one having 2 white marbles and 3 black marbles. One marble is picked from each of the urns. Find the probability that both are white marbles.

Again the answer should be an obvious one: $\frac{5}{11} \times \frac{2}{5} = \frac{2}{11}$.

Now what is the sample space in this case? We learnt that probability was a ratio, $\frac{\text{number of favourable ways}}{\text{total number of ways}}$. In the above question, where did the denominator go?

The sample space in this case is all the following possible outcomes: WW, WB, BW and BB. And when we find the probability of this entire sample space, as expected it will be 1. Check: $\frac{5}{11}\times\frac{2}{5}+\frac{5}{11}\times\frac{3}{5}+\frac{6}{11}\times\frac{2}{5}+\frac{6}{11}\times\frac{3}{5}=\frac{10+15+12+18}{11\times5}=\frac{55}{55}=1$.

Thus, we do not find or write the denominator in such cases.

Now, consider the following problem, with the same scenario...

One marble is picked from each of the urns. If atleast one of the marbles is white, what is the probability that both the marbles are white?

This becomes a question of conditional probability because part information of the outcome is available.

Because of the condition "atleast one of the marble is white", the sample space gets reduced to a limited case, namely WW, WB and BW.

Thus, now the required probability will be

$$\frac{WW}{WW+WB+BW}=\frac{\frac{5}{11}\times\frac{2}{5}}{\frac{5}{11}\times\frac{2}{5}+\frac{5}{11}\times\frac{3}{5}+\frac{6}{11}\times\frac{2}{5}}=\frac{10}{10+15+12}=\frac{10}{37}.$$

Thus, in case of conditional probability, the sample space gets reduced to a smaller, limited number of possibilities. One more example...

E.g. 5: In an oil-field, Reliance, Cairns and Shell are independently exploring for oil. The probability that Reliance strikes oil is 1/3, that Cairns strikes oil is 1/4 and that Shell strikes oil is 1/5. If oil is struck in the oil-field, find the probability that two of the companies struck oil.

This is also a conditional probability question because it states that oil is struck. We have already found in the earlier example with same data that the probability of oil being struck is $1-\frac{2}{3}\times\frac{3}{4}\times\frac{4}{5}=\frac{3}{5}$. Now this will be our denominator when we find the conditional probability.

Two companies striking oil could happen in any of the following ways:
$R\,C\,\not{S}\quad R\,\not{C}\,S\quad \not{R}\,C\,S$.

Thus, required probability is $\frac{\frac{1}{3}\times\frac{1}{4}\times\frac{4}{5}+\frac{1}{3}\times\frac{3}{4}\times\frac{1}{5}+\frac{2}{3}\times\frac{1}{4}\times\frac{1}{5}}{\frac{3}{5}}=\frac{4+3+2}{3\times3\times4}=\frac{1}{4}$.

Binomial Distribution

I take 5 shots at the dart-board. The probability of me hitting the bulls-eye in any attempt is 1/4. What is the probability that I hit bulls-eye on the 1st, 3rd and 5th try?

It should not be a problem to find the probability as $\frac{1}{4}\times\frac{3}{4}\times\frac{1}{4}\times\frac{3}{4}\times\frac{1}{4}=\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^2$.

If the probability was required of the event that I hit the bulls-eye on only the first three attempts?

In this case the answer would be $\frac{1}{4}\times\frac{1}{4}\times\frac{1}{4}\times\frac{3}{4}\times\frac{3}{4}$ which is again $\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^2$.

In-fact the probability of me hitting bulls-eye on any three specified attempts, irrespective of which three attempts are specified is $\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^2$.

But what will be the probability of me hitting bulls-eye on any three attempts i.e. the case when the attempts are not specified?

In this case, any of the following ways is acceptable…

HHHMM or HHMHM or HHMMH or HMHHM or so on. Now the total number of such ways would be ${}^5C_3$ i.e. any three positions out of 5 could be hits and other two would be miss. Now the probability of all these individual case would be $\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^2$ and thus the total required probability will be adding this term a total of ${}^5C_3$ times, i.e. ${}^5C_3\times\left(\frac{1}{4}\right)^3\times\left(\frac{3}{4}\right)^2$.

Any such scenario of finding the probability of r success in a total of n attempts is given by ${}^nC_r\times p^r\times(1-p)^{n-r}$, where p is the probability of success in any one attempt.

The ${}^nC_r$ is to factor-in which of the n attempts results in r successes. And the probability of any one such case would be

$$p\times p\times p\times\ldots_{r\ times}\ldots\times(1-p)\times(1-p)\times\ldots_{(n-r)\ times}=p^r\times(1-p)^{n-r}$$

E.g. 6: A dice is rolled a total of 8 times. What is the probability that I roll a number greater than 4 on exactly 5 rolls?

If s denotes a success i.e. rolling greater than 4 and f denotes a failure i.e. rolling 4 or less, then one possible acceptable way could be $s\ s\ s\ s\ s\ f\ f\ f$. However this is not the only way. The 5 s's could possible come in any 5 out of the 8 positions. Thus the total number of acceptable ways is ${}^8C_5$.

And each of these outcome would have a probability of $\left(\frac{2}{6}\right)^5 \times \left(\frac{4}{6}\right)^3$.

Thus required probability is ${}^8C_5 \times \left(\frac{2}{6}\right)^5 \times \left(\frac{4}{6}\right)^3$

E.g. 7: In the above questions what would be the probability of rolling a number greater than 4 on atleast 5 rolls.

Rolling a number greater than 4 on atleast 5 rolls can be expanded to rolling a number greater than 4 on exactly 5 rolls or 6 rolls or 7 rolls or all 8 rolls.

And the required probability would be the addition of all these cases i.e.

$${}^8C_5 \times \left(\frac{2}{6}\right)^5 \times \left(\frac{4}{6}\right)^3 + {}^8C_6 \times \left(\frac{2}{6}\right)^6 \times \left(\frac{4}{6}\right)^2 + {}^8C_7 \times \left(\frac{2}{6}\right)^7 \times \left(\frac{4}{6}\right)^1 + {}^8C_8 \times \left(\frac{2}{6}\right)^8$$

It should be noticed that the last term of the above boils down to just $\left(\frac{2}{6}\right)^8$

which is as expected because rolling a number greater than 4 on each of the 8 rolls is just $\frac{2}{6} \times \frac{2}{6} \times ... \times \frac{2}{6} = \left(\frac{2}{6}\right)^8$.

Questions on binomial distribution could also be coupled with conditional probability...

E.g. 8: A coin is tossed a total of 8 times. If head appears atleast 6 times, what is the probability that head appears on exactly 7 tosses?

Using the funda of conditional probability, the required probability can be thought of as $\frac{\text{7 heads}}{\text{6 heads or 7 heads or 8 heads}}$.

When a coin is tossed the probability of getting a head and of not getting a head is the same i.e. 1/2. Thus in all the terms, the term equivalent to $p^r \times (1-p)^{n-r}$ would be $\left(\frac{1}{2}\right)^8$.

Thus the required probability is

$$\frac{{}^8C_7 \times \left(\frac{1}{2}\right)^8}{{}^8C_6 \times \left(\frac{1}{2}\right)^8 + {}^8C_7 \times \left(\frac{1}{2}\right)^8 + {}^8C_8 \times \left(\frac{1}{2}\right)^8} = \frac{{}^8C_7}{{}^8C_6 + {}^8C_7 + {}^8C_8} = \frac{8}{28+8+1} = \frac{8}{37}$$

Exercise

9. The probability of A, B and C solving a problem is $\frac{2}{3}, \frac{2}{5}, \frac{3}{8}$. If the problem is solved what is the probability that all three solve the problem?

(1) 7/8 (2) 1/10 (3) 4/35 (4) 4/5

10. Two cards are drawn at random from a well shuffled pack of cards. If the two cards are red cards, find the probability that both the cards are face cards.

(1) 5/442 (2) 3/65 (3) 5/22 (4) 66/325

11. Urn 1 contains 5 white balls and 8 black balls and Urn 2 contains 7 white balls and 2 black balls. One ball is picked from urn 1 and dropped in urn 2. Then one ball is picked from urn 2. If this ball turns out to be white, find the probability that the ball taken from urn 1 and dropped in urn 2 is also white.

(1) 5/12 (2) 7/12 (3) 1/2 (4) 5/13

12. The police inspector fires 10 shots at the terrorist. If the probability that he hits the terrorist on any attempt is 1/7, find the probability that the terrorist is hit.

(1) $\frac{1}{7} \times \left(\frac{6}{7}\right)^9$ (2) $\left(\frac{1}{7}\right)^9 \times \frac{6}{7}$ (3) $1 - \left(\frac{1}{7}\right)^{10}$ (4) $1 - \left(\frac{6}{7}\right)^{10}$

13. In the above question, if the terrorist is hit, find the probability that exactly 1 shot hit the terrorist.

(1) $\frac{6^{10}}{7^{10} - 6^{10}}$ (2) $\frac{6^9}{7^{10} - 6^9}$ (3) $\frac{6^9}{7^{10} - 6^{10}}$ (4) $\frac{6^{10}}{7^{10} - 6^9}$

14. Both A and B fire at the target once. The probability of A hitting the target is 1/3 and of B hitting the target is 2/3. If one bullet hits the target, find the probability that A hits the target and B misses.

(1) 1/5 (2) 1/9 (3) 1/7 (4) 1/8

NOTE:

Sometimes probability is expressed in terms of “odds in favour of” and “odds against”. To convert such data into probability, understand the following...

When the odds in favour of an event is $a : b$, the probability of the event occurring is $\frac{a}{a+b}$.

When the odds against an event is $a : b$, the probability of the event occurring is $\frac{b}{a+b}$.

Answer Key

Permutation & Combination

1. 3	2. 2	3. 4	4. 1	5. 3
6. 2	7. 2	8. 1	9. 4	10. 3
11. 3	12. 1	13. 4	14. 2	15. 1
16. 4	17. 3	18. 4	19.i 4	19.ii. 2
19.iii. 2	20. 1	21. 4	22. 3	23. 4
24. 3	25. 1	26. 3	27. 4	28. 1
29. 1	30. 3	31. 2	32. 3	33. 3
34. 4	35. 3	36. 2	37. 3	38. 2
39. 1	40. 4	41. 4	42. 4	43. 1
44. 3	45. 4	46. 2	47. 1	48. 2
49. 4	50. 3	51. 2	52. 1	53. 3
54. 4	55. 1	56. 3	56. 1	57. 2
58. 1	59. 3	60. 3	61. 4	62. 3
62. 3				

Appendix of P&C

1. 6	2. 15	3. 56	4. 3	5. 34
6. 28	7. 6	8. 5		

Probability

1. 3	2. 2	3. 1	4. 4	5. 4
6. 3	7. 3	8. 1	9. 3	10. 2
11. 1	12. 4	13. 3	14. 1	